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Subject: Applied Mechanics

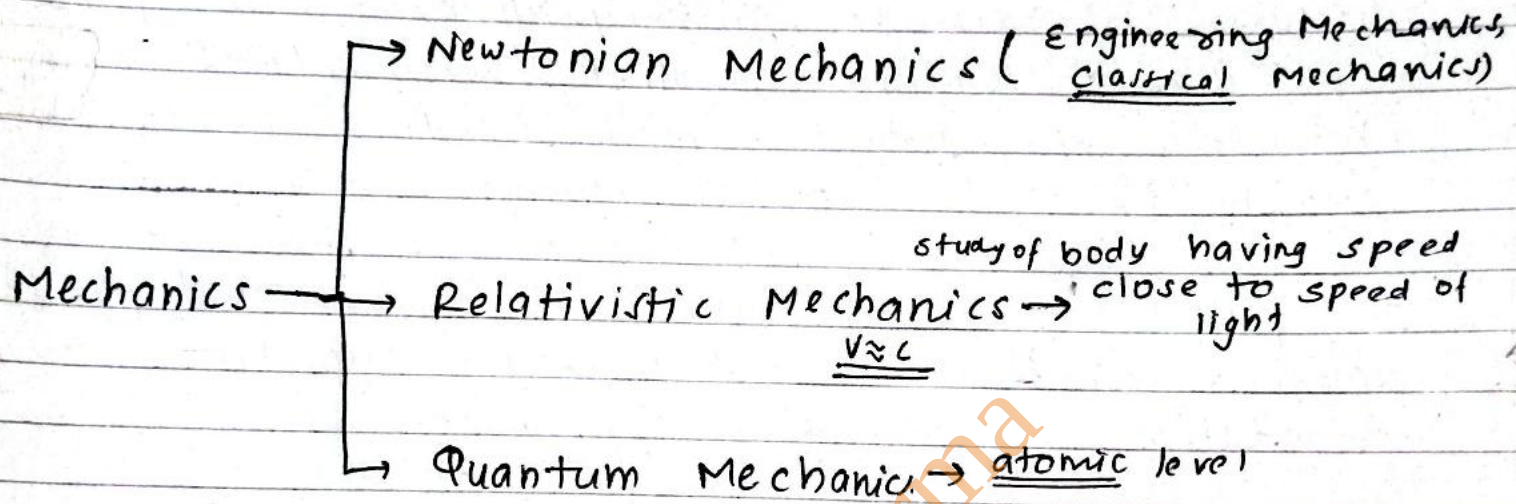
(Subrah Bartola Sir)

Surendra Sharma

Applied Mechanics

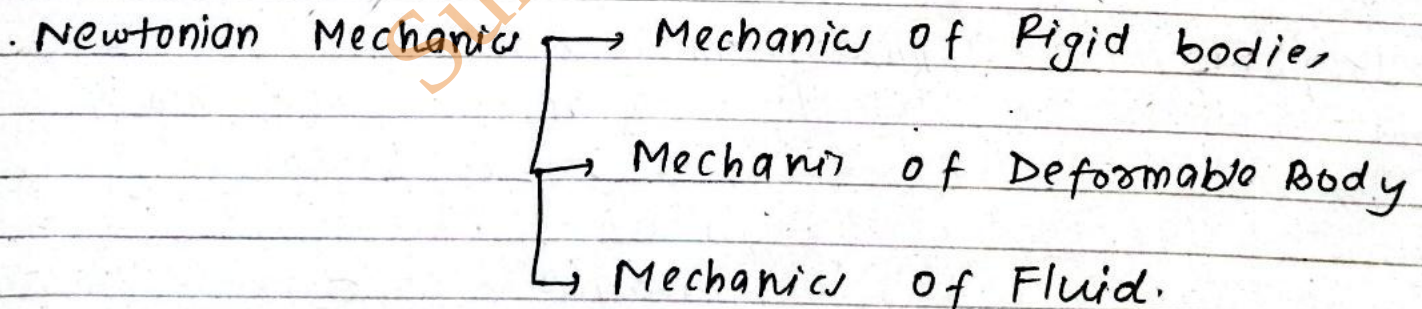
Theory: 20+80

Classification of Mechanics



Mechanics?

↳ It is defined as that science which describes and predicts the condition of rest or motion of bodies under the action of force.



Newtonian Mechanics

↳ Branch of mechanics which deals with the law and principles of mechanics along with their application to engineering problems.

Scope of Applied Mechanics

- The knowledge of engineering mechanics is essential for engineers in planning design and construction of various types of structures and machines. In order to take up his or her job skillfully an engineer must have detailed knowledge of engineering mechanics. In a most systematic and scientific manner.

The purpose of mechanics is to explain and predict physical phenomenon and thus to lay the foundation of engineering application.

Relativistic

Mechanics of Rigid bodies

Statics

Dynamics

Kinetics

Kinematics

Importance of applied mechanics

- engineers are able to plan, design and construct various structures such as towers, bridges, and different civil engineering works.
- Also, design different machines and foundation of machines.

A particle is relative term.

Particle:

→ It is defined as matter which is so small that its dimensions can be treated as negligible in comparison to other dimension involved in problem.

Rigid body

→ It is the combination of two or more particles which are so interconnected that they do not change their relative position however large the force applied maybe.

However, in general all bodies change their shape and size on application of force. But, for solid bodies, it is very small, so are assumed to be perfectly rigid.

Deformable body

Combination of two or more particles which are ~~se~~ interconnected such that, there is appreciable change in shape and size of the body on application of force.

Assumptions made for rigid body:

- The body doesn't deform under the action of applied load.
- Distance b/w any two points of body ^{doesn't change} under application of force.
- We don't have to consider the type of material from which the body is made.

Fluid: Fluid are liquid and gas which do not have their own shape and size rather take the shape and size of container in which it is kept.

Static and Dynamics

Static

↳ Form of mechanics of rigid body which deals with force and their effect ^{while} acting upon the bodies at rest.

Dynamics

↳ Form of mechanics of rigid body which deals with force and their effect while acting upon the ~~the~~ bodies at motion.
↳ Dynamics = $f(t)$

Kinematics

↳ without reference to the cause

Kinetics

↳ with reference to the cause.

Space, time and mass are absolute concepts that are independent of each other in Newtonian mechanics.

Fundamental concepts and principle in mechanics

The basic concepts used in mechanics are

Space, time, mass and force. These concepts cannot be truly defined i.e. these should be accepted on the basis of intuition and experience and used as mental frame of reference for our study of mechanics.

Space:

These are geometric region occupied by bodies whose positions are determined by linear and angular measurement relative to coordinate system.

Time

To define a event, it is not sufficient to indicate its position in space, the time of event should also be given. It is measure of succession of events.

Mass

The concept of mass is used to characterize and compare bodies on the basis of certain fundamental mechanical experiment.

Force

A force represent action of body on another, characterize by magnitude and direction. A point of application.

Fluid: Fluid are liquid and gas which do not have their own shape and size rather take the shape and size of container in which it is kept.

Static and Dynamics

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A force represent action of body on another, characterize by magnitude ~~and~~, direction. & point of application.

The study of mechanics rest upon six fundamental principles.

Fundamental principles

(i) Parallelogram law of addition of forces-

The state

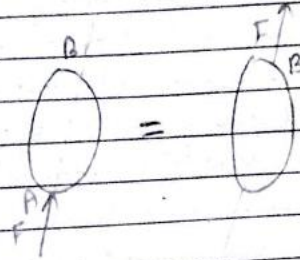
(ii) Principle of Transmissibility.

(iii) Newton's 1st law

(iv) Newton's 2nd law

(v) Newton's 3rd law

(vi) Newton's law of gravitation.



• Principle of Transmissibility.

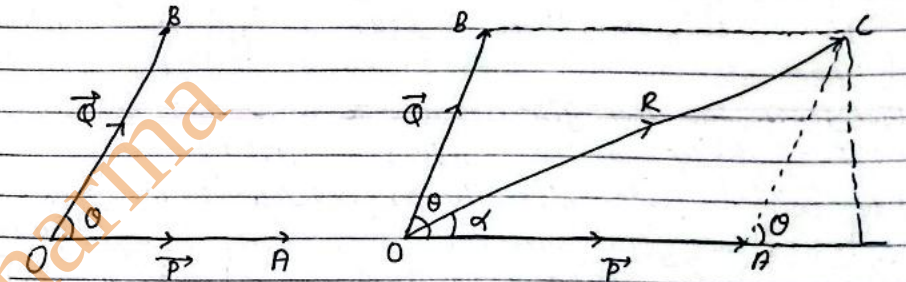
This states that the condition of equilibrium or of motion of rigid body will remain unchanged if a force acting at a given point of the rigid body is replaced by a force of same magnitude and direction but acting at a different point along its same line of action. (provided that the force has same line of action).

Scope of applied mechanics are

- TO describe and predict the condition of rest or motion of bodies under the action of forces.
- For the application of equation of equilibrium to rigid bodies.
- TO know about mechanics of fluids.
- For study of physical phenomena like compressibility.

(i) Parallelogram law of vector addition:-

It states "If two forces acting at a point be represented by magnitude and direction of two adjacent sides of parallelogram, then resultant is represented in magnitude and direction by diagonal of parallelogram passing through that point."



Magnitude of

$$\text{Resultant force (R)} = \sqrt{P^2 + Q^2 + 2PQ \cos \theta}$$

$\theta \rightarrow$ Angle b/w two vectors.

Inclination of resultant with direction of force P:

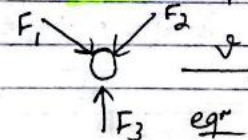
$$\alpha = \tan^{-1} \left(\frac{Q \sin \theta}{P + Q \cos \theta} \right)$$

with $\vec{P}$

(ii) Newton's First Law

"Every body continues in its state of rest or of uniform motion in a straight line if there is no unbalanced force acting upon it"

↳ contains principle of equilibrium of forces



$\vec{F} \rightarrow \vec{a}$ $\vec{F} = m\vec{a}$ $\frac{d\vec{p}}{dt} \propto \vec{F}$
 Accelerated motion

→ basis for most of analysis in Dynamics

(ii) Newton's second law

"The rate of change of linear momentum is directly proportional to the applied force and it takes place in the direction of the applied force."

(iv) Newton's Third Law:

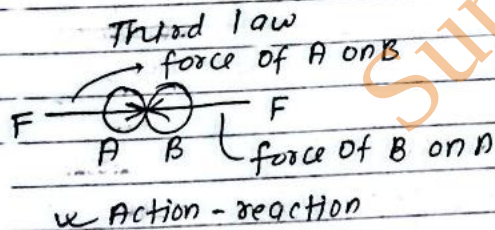
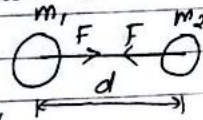
"To every action, there is equal & opposite reaction"
 ↳ force always occur in pairs of equal and opposite forces.

v) Newton's law of gravitation

"Everybody in the universe attracts every other body with a force directly proportional to the product of their mass and inversely proportional to the square of the distance separating them."

↳ governs gravitational attraction b/w two bodies.

$F = \frac{G m_1 m_2}{d^2}$; $G = 6.67 \times 10^{-11} \text{ Nm}^2/\text{kg}^2$
 ↳ Universal Gravitational constant



Examples

(1) First law: A ball rolling down the hill will continue to roll unless friction or another force stops it.

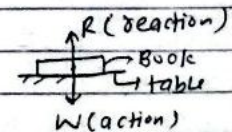
(2) Second law: If a baseball player hits a ball with double the force, the rate at which the ball will accelerate (speed up) will be doubled.

$F_{\text{applied}} = \frac{m_{\text{ball}} \times a_{\text{ball}}}{\text{constant}}$

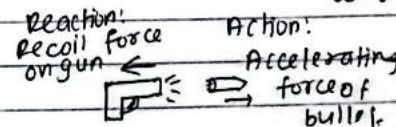
$F_{\text{applied}} \propto a_{\text{ball}}$

(3) Third law: (i) A book lying on a table:

The wt. of book is acting in downward direction on the table (action) while reaction acts in the upward direction on the book.



(ii) During swimming, a swimmer pushes the water back in order to accelerate forward.



Moment of force P about O
 $M_P = 2 \times \text{Area of } \triangle AOB$



$$\triangle AOB = \frac{1}{2} P \times d$$

$$M = 2 \times \triangle AOB$$

Varignon's Theorem: vertex moment center Base $\rightarrow$ force

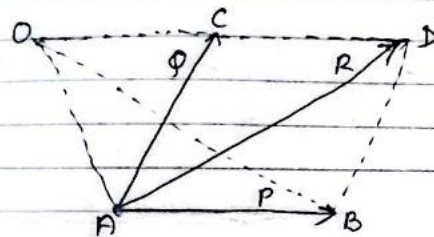
concurrent

"Moment of a resultant of two forces, about a point lying in the plane of the forces, is equal to the algebraic sum of moments of these forces about the same point."

Two concurrent forces P & Q

$R \rightarrow$ Resultant force

O $\rightarrow$ point about which moment is to be calculated.



Now,

Moment of force P about O = $2 \times \triangle AOB$

Moment of force Q about O = $2 \times \triangle AOC$

" " R about O = $2 \times \triangle AOD$

From Geometry

$$\triangle AOD = \triangle AOC + \triangle ACD$$

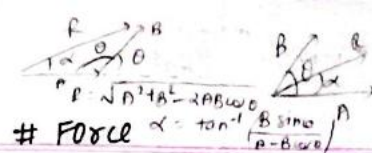
$$= \triangle AOC + \triangle ABD$$

$$= \triangle AOC + \triangle AOB \quad \{ \triangle AOB \text{ \& } \triangle ABD \text{ have identical base \& ht.} \}$$

$$\therefore \triangle AOD = \triangle AOC + \triangle AOB$$

$$\text{or, } \frac{M_R}{2} = \frac{M_Q}{2} + \frac{M_P}{2}$$

$$\therefore M_R = M_Q + M_P$$



$$R = \sqrt{P^2 + Q^2 + 2PQ \cos \theta}$$

$$\alpha = \tan^{-1} \left(\frac{P \sin \theta}{P + Q \cos \theta} \right)$$

Force $\alpha = \tan^{-1} \left(\frac{P \sin \theta}{P + Q \cos \theta} \right)$

$\rightarrow$ The quantitative description of interaction of two physical bodies is termed as force.
 $\rightarrow$ Concept of force is given by Sir Isaac Newton.

Characteristics of Force:

- Force is a vector quantity.
- It should have point of application. (contact point b/w two bodies).
- It ~~has the~~ can be moved along its line of action.

* Classification:

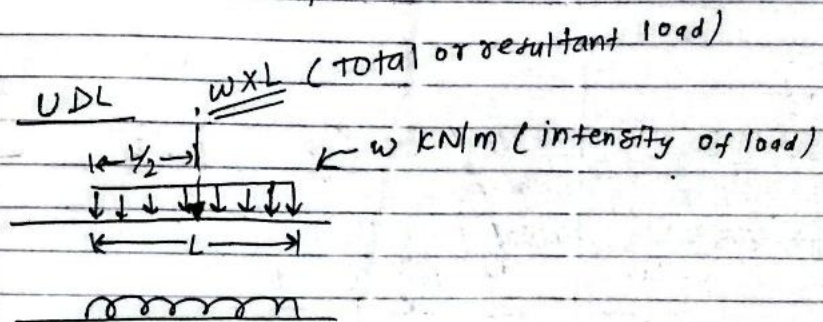
(i) General Classification:

- External force $\rightarrow$ Contact force
 $\rightarrow$ Body force. (eg: Gravitation)
- Internal force

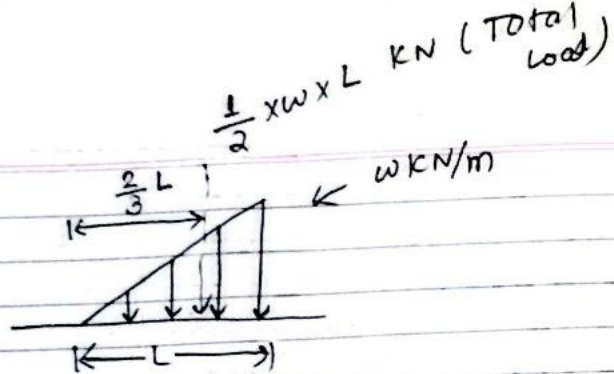
(ii) With respect to area:

- Distributed force $\rightarrow$ Uniform $\rightarrow$ Uniformly distributed force (load) [UDF]
 $\rightarrow$ Non-uniform $\rightarrow$ Uniformly varying force (UVL)

(ii) Concentrated force



UVL



(3) ~~at~~ with respect to based on ^{effect of} force

(i) Translational force

(ii) Rotational force

Few more types of forces:

(A) Collinear Force: same line of action.

(B) Concurrent Force: passing through same point

(C) Coplanar Force: ~~are~~ lying on same plane.

(D) Coplanar and concurrent

(E) Parallel Force: same direction

Common vector addition

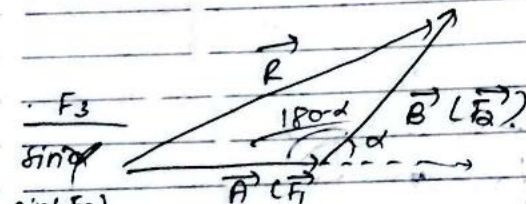
(1) Triangle law of vector addition. (Triangle Rule)

(i) Law of cosines: $R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2 \cos \alpha}$

(ii) Law of sines

↓

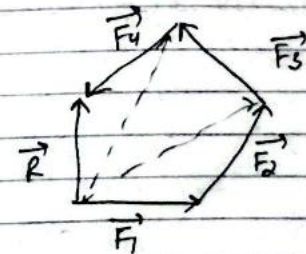
$$\frac{F_1}{\sin(\alpha)} = \frac{F_2}{\sin(\beta)} = \frac{F_3}{\sin(\gamma)}$$



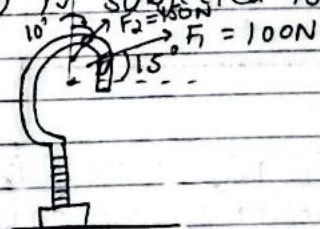
$$\vec{R} = \vec{A} + \vec{B}$$

(2) Polygon law of vector addition.

$$\vec{R} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4$$



Q. The screw eye (hook) is subjected to two forces F_1 and F_2 as shown

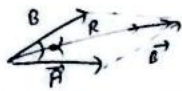


Determine the magnitude and direction of resultant

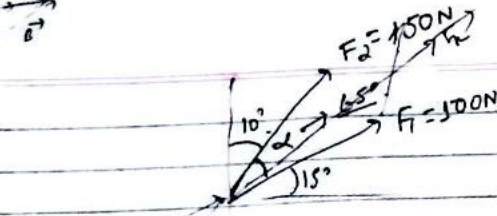
using (i) parallelogram law of vector addition

(ii) Vector triangle.

Further, find the.



$$\vec{R} = \vec{A} + \vec{B} \quad \vec{A} \cdot \vec{B} = AB \cos \theta$$



$$R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2 \cos \alpha}$$

$$= \sqrt{100^2 + 150^2 + 2 \times 100 \times 150 \times \cos 65^\circ}$$

$$= 212.55 \text{ N}$$

Let, the Resultant vector makes β angle with $\vec{F}_1$.

$$\tan \beta = \frac{F_2 \sin \alpha}{F_1 + F_2 \cos \alpha}$$

$$= \frac{150 \times \sin 65^\circ}{100 + 150 \times \cos 65^\circ}$$

$$\beta = 39.76^\circ \text{ with } \vec{F}_1$$

$$\text{ \& } 54.76^\circ \text{ with } x\text{-axis}$$

FBD helps to visualize force and moments applied to a body and to calculate the resulting reactions in many types of problem in mechanics

Simple sketch of body/object which contains applied forces, on forces and moments. ~~It is called FBD~~ Space dia
 → supports & connections are removed & drawn.

Free body diagram

In practice, a problem in engineering mechanics is derived from an actual physical situation. A sketch showing a physical condition of the problem is known as space diagram. A free body diagram is a space diagram in which significant particle or body is chosen and a separate diagram is drawn showing ^{the particle} all the force acting on it. The forces may be reaction force, applied force and moment.

Guidelines to draw FBD

(i) → Selection of portion

→ Removal of external

→ support and connections

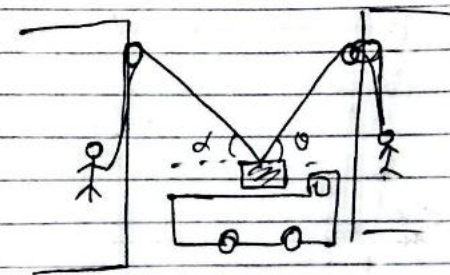
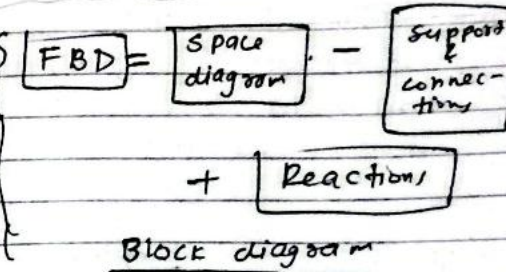
→ Coordinate system

→ Dimensions Indication along with slope.

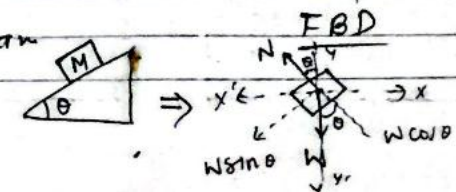
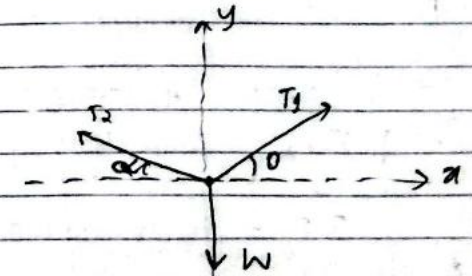
→ Applied load (including weight)

→ Indication with an arrow along with labeling.

→ Reaction indication with arrow along with labeling.

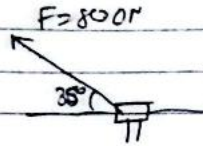


Space diagram

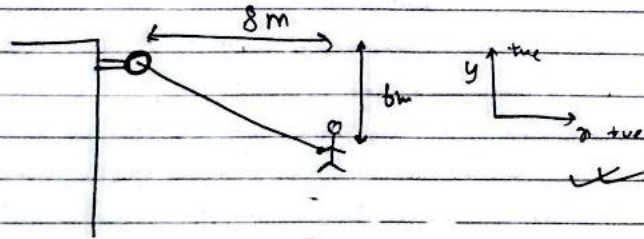


Rectangular component of forces.

Q.1 A force of 800N is exerted on a bolt as shown. Determine the horizontal and vertical component of force.



Q.2 A man pulls with a force of 300N on a rope attached to a building. What are the horizontal and vertical component of force exerted by a rope at point A.



Imp.

Q.4 Use rectangular component of forces to determine the solution to Q.N.1

Split all forces in ver. & hor. direction (co-ordinate system)

ΣF_x = summation of all forces in x-direction (horizontal direction)

ΣF_y = summation of

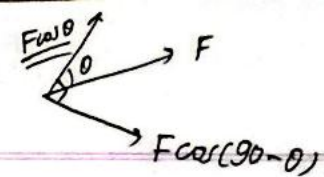
$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$$

$$\tan \theta = \frac{\Sigma F_y}{\Sigma F_x}$$

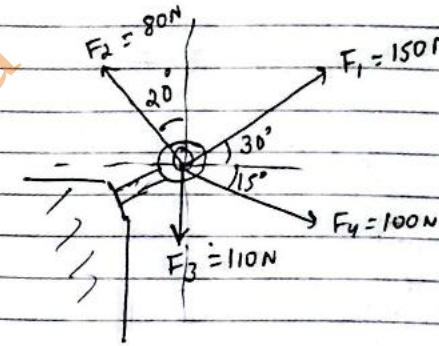
(with x-axis)

$$\tan \alpha = \frac{\Sigma F_y}{\Sigma F_x}$$

(with x-axis)



Q.5 Four forces act on a bolt, determine the resultant of the forces on the bolt.



Importance of FBD:

- Helps to visualize all the forces acting on a single object.
- Helps to solve complex problem.
- Helps to determine numerical value of the reaction, that occurs at support and hence internal forces i.e. axial force, shear force and BM can be easily calculated.

Static equilibrium

A particle is said to be in static equilibrium if the resultant of all the forces and moments acting on it is zero. Hence, when the particle is in equilibrium, the net effect of given force is zero.

Mathematically,

$$\begin{aligned} \text{Particle} &\leftarrow \Sigma F = 0 & \{ \Sigma F_x = 0, \Sigma F_y = 0 \} \\ \text{System of particle} &\leftarrow \Sigma M = 0 & \{ \Sigma M_x = 0, \Sigma M_y = 0, \Sigma M_z = 0 \} \end{aligned}$$

2D → for different planes.

In-3D (xyz space)

$$\Sigma F_x = 0$$

$$\Sigma F_y = 0$$

$$\Sigma F_z = 0$$

$$\Sigma M_x = 0$$

$$\Sigma M_y = 0$$

$$\Sigma M_z = 0$$

In 2D (Coplanar forces)

For xy plane

$\Sigma F_x = \text{Summation of forces along x-dir} = 0$

$$\Sigma F_y = 0$$

$\Sigma M \text{ about z-axis} = 0$ { For particle non-significant as forces acting on it will be concurrent }

For yz plane

$$\Sigma F_x = 0$$

Graphical representation (Physical meaning)

(i) For a particle

A particle acted upon by 2 forces will be in equilibrium if two forces have same magnitude, same line of action but opposite sense (direction).

$$\begin{aligned} &\xrightarrow{F_1} \quad \quad \quad \xleftarrow{F_2} \\ &F_1 = F_2 \end{aligned}$$

i.e. the resultant of two forces is 0

2. For Number of forces

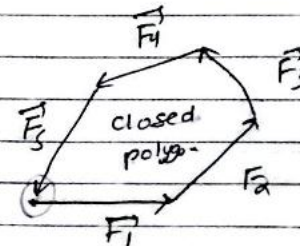
For a particle acted upon by more than two forces, polygon rule is used.

In this case,

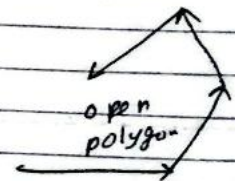
starting from one force and ~~at the~~ arranging all the forces in tip-to-tail fashion, a polygon is drawn. If, tip of last force drawn coincides with the tail of first force drawn, then the particle is in equilibrium.

i.e.

A closed polygon drawn provides the graphical expression of equilibrium.

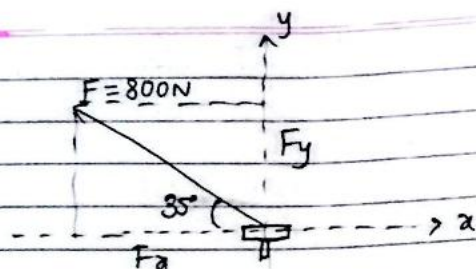


In equilibrium.



Not in equilibrium

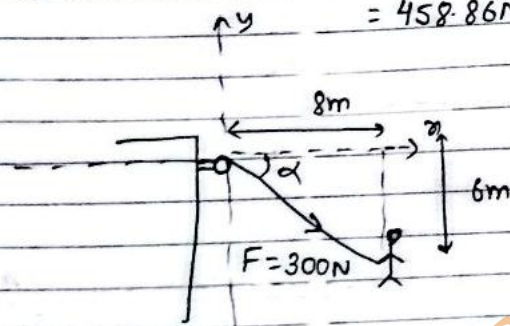
Q.2



Here,
Horizontal component of force (F_x) = $-F \cos 35^\circ$
 $= -800 \times \cos 35^\circ$
 $= -655.32 \text{ N}$

Vertical component of force (F_y) = $F \sin 35^\circ$
 $= 800 \times \sin 35^\circ$
 $= 458.86 \text{ N}$

Q.3



Here,
Let, α be the angle made by horizontal with the pulled force.

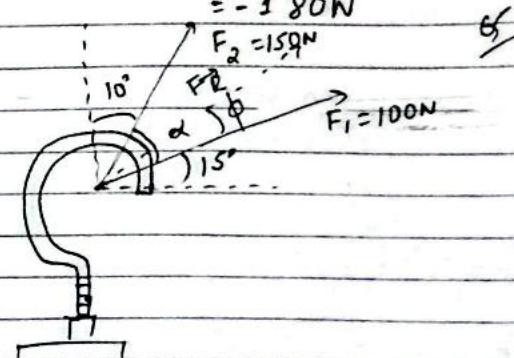
$$\tan \alpha = \frac{6}{8}$$

$$\Rightarrow \alpha = 36.87^\circ$$

Horizontal component of force (F_x) = $F \cos \alpha$
 $= 300 \times \cos(36.87^\circ)$
 $= 239.99 \text{ N}$

Vertical component of force (F_y) = $-F \sin \alpha$
 $= -300 \sin(36.87^\circ)$
 $= -180 \text{ N}$

Q.4



Here,

$$\Sigma F_x = F_1 \cos 15^\circ + F_2 \sin 10^\circ = 122.64 \text{ N}$$

$$\Sigma F_y = F_1 \sin 15^\circ + F_2 \cos 10^\circ = 173.6 \text{ N}$$

Now,

$$\text{Resultant force (R)} = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$$

$$= \sqrt{(122.64)^2 + (173.6)^2}$$

$$= 212.55 \text{ N}$$

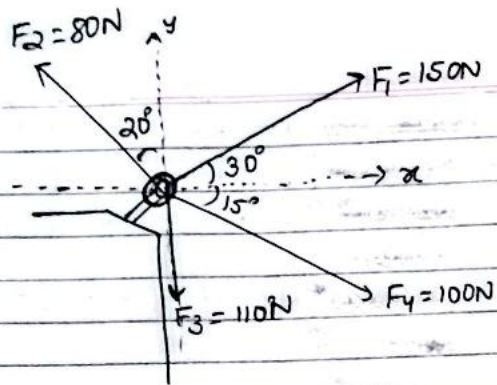
$\theta \rightarrow$ Angle made with x-axis,

$$\tan \theta = \frac{\Sigma F_y}{\Sigma F_x} = \frac{173.6}{122.64}$$

$$\theta = 54.76^\circ$$

with x-axis

Q.5



Here,

$$\begin{aligned}\Sigma F_x &= F_1 \cos 30^\circ + F_4 \cos 15^\circ - F_2 \sin 20^\circ \\ &= 150 \cos 30^\circ + 100 \cos 15^\circ - 80 \sin 20^\circ \\ &= 199.134 \text{ N}\end{aligned}$$

$$\begin{aligned}\Sigma F_y &= F_1 \sin 30^\circ + F_2 \cos 20^\circ - F_4 \sin 15^\circ - F_3 \\ &= 14.3 \text{ N}\end{aligned}$$

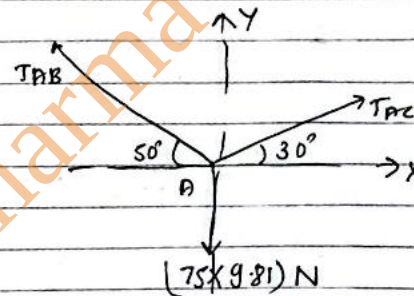
$$\begin{aligned}\text{Resultant force on bolt (R)} &= \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2} \\ &= 199.64 \text{ N.}\end{aligned}$$

$$\begin{aligned}\alpha &= \tan^{-1} \left(\frac{14.3}{199.134} \right) \\ &= 4.1^\circ\end{aligned}$$

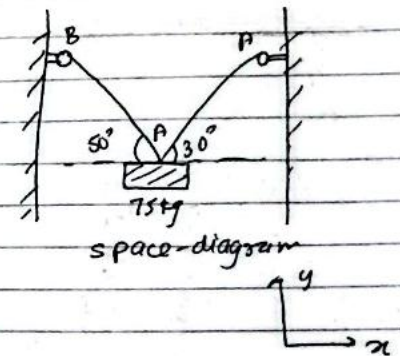
✓ Force for max. Resultant
✓ forces which gives resultant at given angle.

Q.6

A 75kg crate shown in the space diagram was laying b/w two buildings and it is being lifted onto a Truck which will remove it. The crate is supported by a vertical cable which is joined at 'A' to two ropes which pass over pulley attached to a building at 'B' and 'C'. Determine the tension in each rope AB and AC.



$$\Sigma F = 0$$



$$\begin{aligned}\ast \Sigma F_x &= 0 \quad (\rightarrow +) \\ T_{AC} \cos 30^\circ - T_{AB} \cos 50^\circ &= 0 \quad \text{--- (i)}\end{aligned}$$

$$\begin{aligned}\ast \Sigma F_y &= 0 \quad (\uparrow +) \\ T_{AC} \sin 30^\circ + T_{AB} \sin 50^\circ &= 75 \times 9.81 \quad \text{--- (ii)}\end{aligned}$$

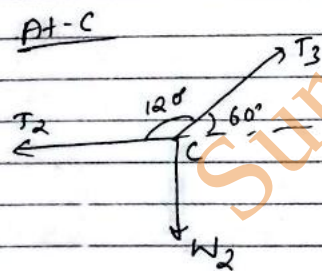
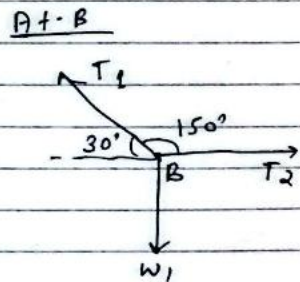
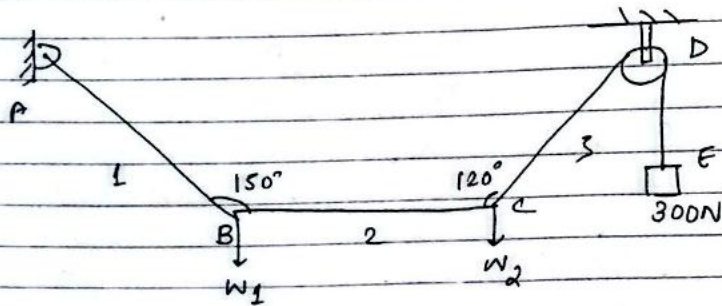
Solving (i) & (ii),

$$T_{AC} = 480.23 \text{ N}$$

$$T_{AB} = 647 \text{ N}$$

or Lami's Theorem

Q.7A light string ABCD whose extremity A is fixed has weight W_1 and W_2 attached to it at B and C. It passes through a small smooth peg at D carrying a wt. of 300N at the free end E as shown. If in the eq^m position shown BC is horizontal and AB and CD make 150° & 120° with BC. Find tension string in the string and weight.



$T_{DE} = T_3$ {Frictionless smooth pulley}
 $\therefore T_3 = 300\text{N}$ — (1) ✓

For C

$\sum F_x = 0 \Rightarrow T_2 = T_3 \cos 60 = 150\text{N}$ ✓

$\sum F_y = 0 \Rightarrow W_2 = T_3 \sin 60 = 259.8\text{N}$ ✓

For C

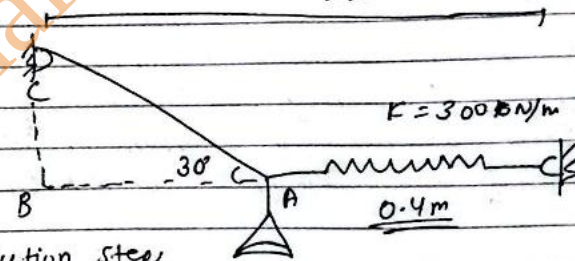
$T_2 = T_1$

$\sum F_x = 0 \Rightarrow T_1 \cos 30 = T_2$

$T_1 = 173.2\text{N}$ ✓

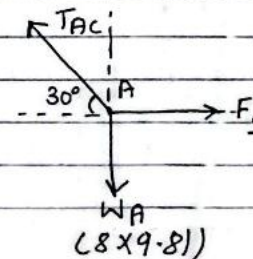
$\sum F_y = 0 \Rightarrow W_1 = T_1 \sin 30 = 86.6\text{N}$ ✓

Q.8 Determine the required length of cord AC so that the 8kg lamp can be suspended in the position shown. The undeformed length of the spring is 0.4m and the spring has stiffness 300N/m .



Solution steps

- (i) Draw FBD at A
- (ii) Use eq^m equations to find T_{AC} and T_{AB}
- (iii) Determine stretch using $F = k\Delta x$
- (iv) Determine length of AC using geometry.



For A to be in equilibrium,

$T_{AC} \sin 30 = W_A$

$\sum F_y = 0 \Rightarrow T_{AC} = 156.96\text{N}$

Also, $\sum F_x = 0$

$F_1 = T_{AC} \cos 30 = 135.93\text{N}$

F_s is the force exerted by stretched string.

So,

Force exerted by stretched string (F_s) = kx

$$\Rightarrow x = \frac{F_s}{k} = \frac{135.93}{300}$$

$$\text{extension}(x) = 0.453 \text{ m}$$

$$AB = 2 - 0.4 - 0.453 \Rightarrow AB = 1.147 \text{ m}$$

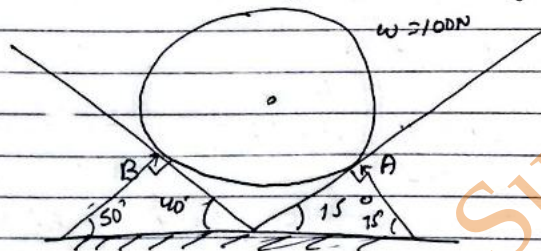
unextended length extended length

$$\text{length of AC} = \frac{AB}{\cos 30^\circ} = \frac{1.147}{\cos 30^\circ} = 1.324 \text{ m}$$

Q.9 A smooth circular cylinder of radius 1.5m is lying in a triangular groove one side of which makes 15° angle and other 40° angle with horizontal. Find rxn at the surface of contact if there is no friction and the cylinder weigh 100N.

$$R_A = 78.5 \text{ N}$$

$$R_B = 31.6 \text{ N}$$



Q.10 Solve Q.N.9 using Lami's Theorem.

→ Solution,

Free-body diagram (FBD) of cylinder.

Under equilibrium condition,

$$\Sigma F_x = 0 \quad \text{--- (i) } (\rightarrow +ve)$$

$$R_B \cos 50^\circ - R_A \cos 75^\circ = 0 \quad \text{--- (i)}$$

$$\Sigma F_y = 0 \quad (\uparrow +ve)$$

$$R_B \sin 50^\circ + R_A \sin 75^\circ = 100 \quad \text{--- (ii)}$$

Solving (i) & (ii)

$$R_B = 31.6 \text{ N}$$

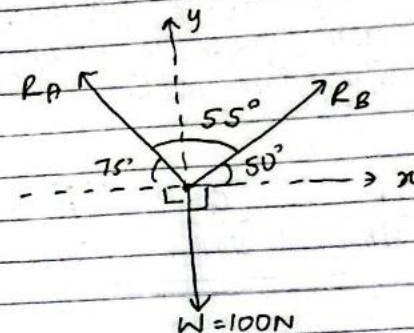
$$R_A = 78.47 \text{ N}$$

Using Lami's Theorem,

$$\frac{R_A}{\sin(140^\circ)} = \frac{R_B}{\sin(165^\circ)} = \frac{W(100)}{\sin(55^\circ)}$$

$$R_A = 78.47 \text{ N}$$

$$R_B = 31.6 \text{ N}$$

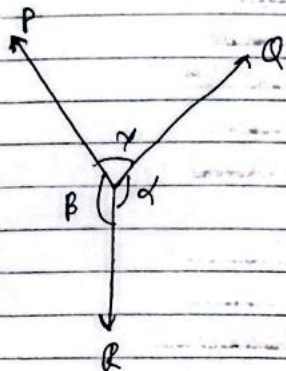


~ cylinder / sphere

Lami's Theorem

It states that, "If three coplanar forces acting at a point be in equilibrium then each force is proportional to the sine of angle b/w the other two."

$$\frac{P}{\sin \alpha} = \frac{Q}{\sin \beta} = \frac{R}{\sin \gamma}$$



Q.11 Two spheres A and B of wt. 100N and 150N and radius 30cm and 40cm respectively are placed in a vessel. Find the rxn at the contact point. Base length is 1.2m.

→ solution,

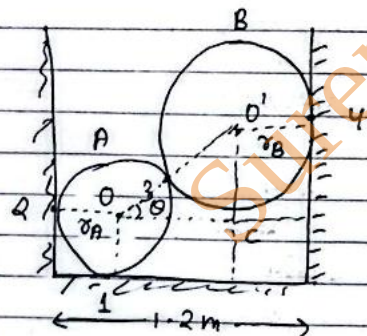
$$OC = 1.2 - r_A - r_B = 1.2 - 0.3 - 0.4$$

$$OC = 0.5m$$

$$OO' = r_A + r_B = 0.3 + 0.4 = 0.7$$

$$\theta = \cos^{-1} \left(\frac{OC}{OO'} \right) = \cos^{-1} \left(\frac{0.5}{0.7} \right)$$

$$\theta = 44.42^\circ$$



FBD of B

Under stable equilibrium:-

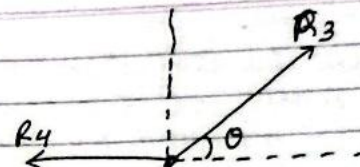
$$\Sigma F_y = 0 \quad (\uparrow +ve)$$

$$\Rightarrow R_3 \sin \theta = W_B$$

$$\Rightarrow R_3 = \frac{150}{\sin 44.42^\circ} = 214.31N \quad W_B = 150N$$

$$\Sigma F_x = 0 \quad (\rightarrow +ve)$$

$$R_4 = R_3 \cos \theta = R_3 \cos(44.42^\circ) = 153.068N$$



FBD of A

Under stable equilibrium,

$$\Sigma F_y = 0 \quad (\uparrow +ve)$$

so,

$$R_1 = W_A + R_3 \sin \theta$$

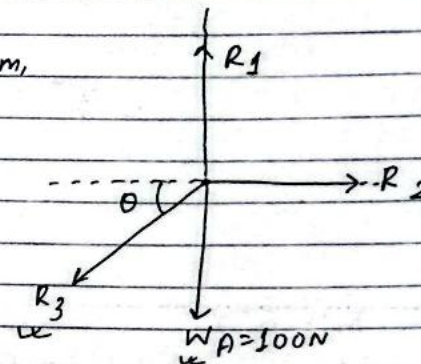
$$= 100 + 214.31 \sin(44.42^\circ)$$

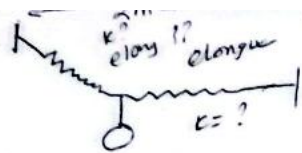
$$R_1 = 250N$$

$$\Sigma F_x = 0 \quad (\rightarrow +ve)$$

$$R_3 \cos \theta = R_2 = R_3 \cos \theta$$

$$\text{or, } R_3 = R_2 \quad R_2 = 214.31 \times \cos(44.42^\circ) = 153.068N$$





Moment of force about a point

↳ Moment of force about a point or axis is measure of the tendency of force to rotate a body about a point or axis

$$M_o = F \times d$$

↳ d is distance b/w force & point

Vectorially,

$$\vec{M} = \vec{r} \times \vec{F}$$

Scalar & Vector quantities & their properties.

$$\vec{M} = \vec{r} \times \vec{F}$$

If, $\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$

$$\vec{r} = r_x \hat{i} + r_y \hat{j} + r_z \hat{k}$$

Then,

$$\vec{M} = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

Note: Anticlockwise moment $\rightarrow$ +ve
clockwise moment $\rightarrow$ -ve

MOF doesn't characterize the point of application of force along its line of action of force. $\rightarrow$ Twisting

↳ Direction of moment is given by right-hand thumb rule

Moment of force about a point

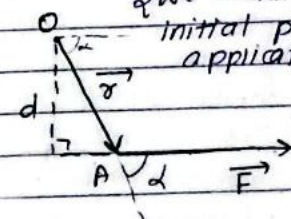
$$\vec{M}_o = \vec{r} \times \vec{F}$$

$$= r F \sin \alpha \hat{n}$$

But $r \sin \alpha$ equals to the $\perp$ distance ' d ' from line of action of force to the moment centre O .

↳ The distance ' d ' is independent of position A along the line of action of force.

↳ We usually consider initial point of application of force



i.e.

$$r_1 \sin \alpha_1 = r_2 \sin \alpha_2 = d$$

$$\therefore \vec{M}_o = Fd \hat{n}$$

Case I:

$$\begin{aligned} \textcircled{I} \quad M_1 &= \vec{r} \times \vec{F} \\ &= r_1 F_1 \sin \alpha_1 \\ &= Fd \end{aligned}$$

Case II:

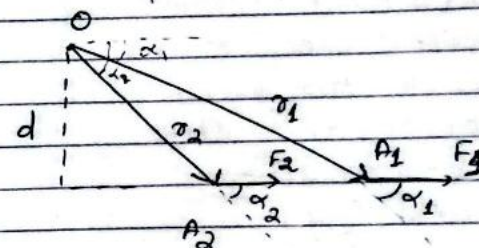
$$\begin{aligned} \textcircled{II} \quad M_2 &= \vec{r} \times \vec{F} \\ &= r_2 F_2 \sin \alpha_2 \\ &= Fd \end{aligned}$$

Thus, $M_1 = M_2$

A_1, A_2
Point of application of force F_1 & F_2 .

But, $A_1 \Rightarrow F_1$
 $A_2 \Rightarrow F_2$

$$F_1 = F_2 = F$$



The Moment of force about a point depends upon a magnitude, line of action, and the sense of the force but doesn't depend upon the actual position of application of force, along the line of action of force.

i.e.

Moment (M_0) of a force F doesn't ~~the~~ characterize the position of point of application of force along the line of action.

Moment of force about an axis

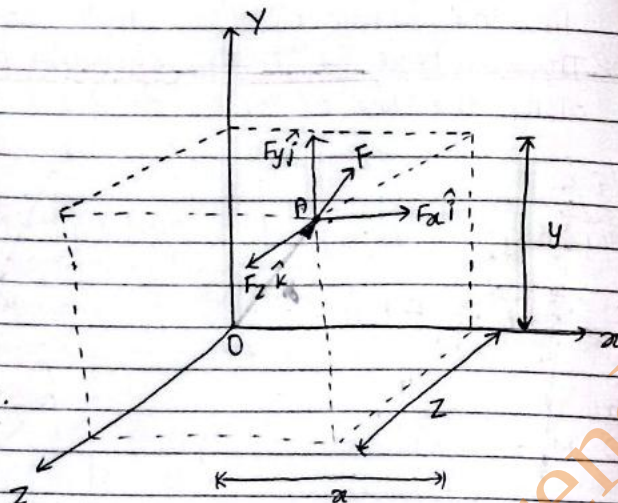
Let,

$$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$$

$$\vec{r} = x \hat{i} + y \hat{j} + z \hat{k}$$

Conceptually,

To find moment about x -axis, we see that component $F_x \hat{i}$ has no turning effect about x -axis. While turning effect of $F_y \hat{j} + F_z \hat{k}$ about x -axis are $-F_y z$ and $F_z y$.



Mathematically,

Moment of $\vec{F}$ about a point O

$$\vec{M}_O = \vec{r} \times \vec{F}$$

$$= (F_z y - F_y z) \hat{i} + (F_x z - F_z x) \hat{j} + (F_y x - F_x y) \hat{k}$$

Now,

$$\vec{M} \cdot \hat{i} = (F_z y - F_y z) = \text{Moment of force about } x\text{-axis}$$

Similarly,

$$\vec{M} \cdot \hat{j} = F_x z - F_z x = \text{Moment of force about } y\text{-axis}$$

$$\vec{M} \cdot \hat{k} = F_y x - F_x y = \text{Moment of force about } z\text{-axis.}$$

→ In general,

✓ Moment of F about an axis AB equal $\vec{M} \cdot \hat{n}$

where, $\vec{M}$ is moment of force about any point on axis AB

$\hat{n}$ is unit vector along AB .

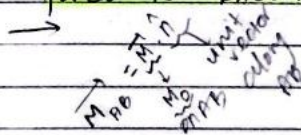
Principle of moments:-

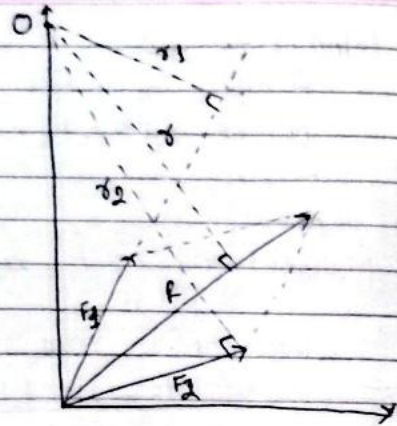
Moment (M) of the resultant (R) of a system of forces with respect to any axis or point is equal to vector sum of moments of individual forces of the system with respect to same axis or point.

i.e.

$$\vec{r} \times \vec{R} = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \dots$$

Application of this principle to pair of concurrent forces is known as Varignon's theorem.

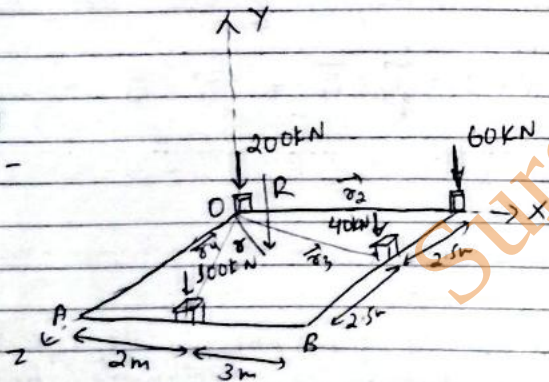




Q. A square mat foundation supports four columns as shown. Determine the magnitude and the point of application of the resultant loads.

⇒ Solⁿ:-

$$\begin{aligned} \text{Resultant forces } (F_R) &= -200\hat{j} - 60\hat{j} - 40\hat{j} - 100\hat{j} \\ &= -400\hat{j} \end{aligned}$$



Let, resultant force act at a point (x, y, z) with respect to O.

$\vec{r}$	$\vec{F}$	$\vec{r} \times \vec{F}$
$0\hat{i} + 0\hat{j} + 0\hat{k}$	$-200\hat{j}$	0
$5\hat{i}$	$-60\hat{j}$	$-300\hat{k}$
$5\hat{i} + 2.5\hat{k}$	$-40\hat{j}$	$-200\hat{k} + 100\hat{i}$
$2\hat{i} + 5\hat{k}$	$-100\hat{j}$	$-200\hat{k} + 500\hat{i}$
Resultant	$\vec{F}_R$	$\Sigma \vec{r} \times \vec{F}$
	$\Sigma \vec{F} = -400\hat{j}$	$\Sigma (\vec{r} \times \vec{F}) = 600\hat{i} - 700\hat{k}$

From principle of moment,

$$\vec{r} \times \vec{F}_R = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \vec{r}_3 \times \vec{F}_3 + \vec{r}_4 \times \vec{F}_4$$

$$\text{or, } (x\hat{i} + y\hat{j} + z\hat{k}) \times (-400\hat{j}) = (0\hat{i} + 0\hat{j} + 0\hat{k}) \times (-200\hat{j}) + (5\hat{i}) \times (-60\hat{j}) + (5\hat{i} + 2.5\hat{k}) \times (-40\hat{j}) + (2\hat{i} + 5\hat{k}) \times (-100\hat{j})$$

$$\text{or, } -400x\hat{k} + 400z\hat{i} = 0 - 300\hat{k} - 200\hat{k} + 100\hat{i} - 200\hat{k} + 500\hat{i}$$

Thus, Equating like coeff.

$$-400x = -700$$

$$x = \frac{7}{4} = 1.75$$

$$400z = 600$$

$$z = 1.5$$

Thus,

required point of application is $(1.75, 0, 1.5)$

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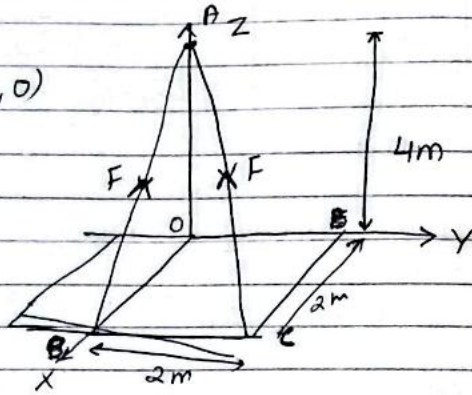
The cantilever roof is supported by cables as shown. If a cable ~~exerts~~ exerts force $F_{AB} = 100\text{N}$ & $F_{AC} = 120\text{N}$ on the wall at A. Determine the resultant force acting at A.

→ Solⁿ:-

$$A(0,0,4), B(2,0,0), C(2,2,0)$$

$$\text{Now, } \hat{AB} = \frac{-2\hat{i} + 0\hat{j} + 4\hat{k}}{\sqrt{20}}$$

$$\hat{AC} = \frac{-2\hat{i} - 2\hat{j} + 4\hat{k}}{\sqrt{24}}$$



$$\text{Now, Resultant force at A } (F_R) = F_{AB}(\hat{AB}) + F_{AC}(\hat{AC})$$

$$= 100\left(\frac{-1}{\sqrt{5}}\hat{i} + \frac{2}{\sqrt{5}}\hat{k}\right) + 120\left(\frac{-1}{\sqrt{6}}\hat{i} - \frac{1}{\sqrt{6}}\hat{j} + \frac{2}{\sqrt{6}}\hat{k}\right)$$

Resultant force at A:

$$\vec{F}_R = \vec{F}_{AB} + \vec{F}_{AC} = F_{AB}(\hat{AB}) + F_{AC}(\hat{AC})$$

$$\vec{F}_R = 100\left(\frac{-1}{\sqrt{5}}\hat{i} + \frac{2}{\sqrt{5}}\hat{k}\right) + 120\left(\frac{-1}{\sqrt{6}}\hat{i} - \frac{1}{\sqrt{6}}\hat{j} + \frac{2}{\sqrt{6}}\hat{k}\right)$$

$$\vec{F}_R = 93.72\hat{i} - 48.98\hat{j} + 187.43\hat{k}$$

$$\vec{F}_R = 93.72\hat{i} - 48.98\hat{j} + 187.43\hat{k}$$

Q14 Two forces act on a hook specify the magnitude of F_2 . Specify the magnitude of F_2 and its coordinate direction angle such that resultant for F_R acts along the positive ~~x~~ y-axis and has a magnitude of 800N. ($F_2 = 700\text{N}$)

→ Solⁿ:-

Here,

$$\text{For } \vec{F}_1, |\vec{F}_1| = 300\text{N}$$

$$\alpha = 45^\circ, \beta = 60^\circ, \gamma = 120^\circ$$

Now,

$$\vec{F}_1 = F_1 \cos \alpha \hat{i} + F_1 \cos \beta \hat{j} + F_1 \cos \gamma \hat{k}$$

$$= 212.13\hat{i} + 150\hat{j} - 150\hat{k}$$

$$\vec{F}_R = 800\hat{j} \text{ (Resultant force)}$$

Let, ~~α, β, γ~~ be angles made by $\vec{F}_2$ with x-axis, y-axis & z-axis respectively.

$$\text{Let, } \vec{F}_2 = F_2 \hat{i} + F_2 \hat{j} + F_2 \hat{k}$$

As we know,

$$\vec{F}_R = \vec{F}_1 + \vec{F}_2$$

$$\text{or } 800\hat{j} = 212.13\hat{i} + 150\hat{j} - 150\hat{k} + F_2\hat{i} + F_2\hat{j} + F_2\hat{k}$$

$$\text{or } F_2\hat{i} + F_2\hat{j} + F_2\hat{k} = -212.13\hat{i} + 650\hat{j} + 150\hat{k}$$

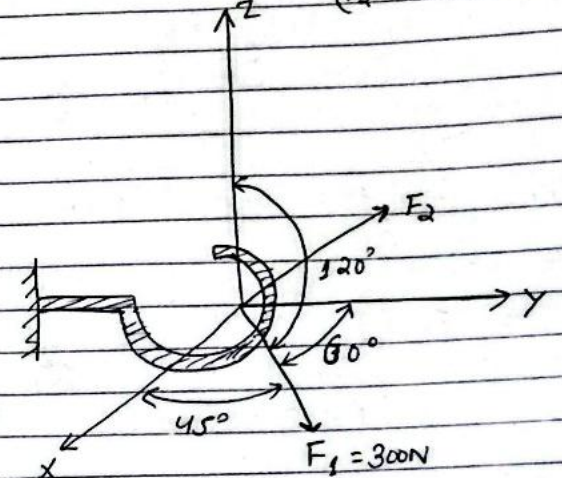
$$|\vec{F}_2| = \sqrt{F_x^2 + F_y^2 + F_z^2} = \sqrt{(-212.13)^2 + (650)^2 + (150)^2} = 700\text{N}$$

Let, α', β', γ' be angles made by $\vec{F}_2$ with x, y & z-axis resp.

$$\alpha' = \cos^{-1}\left(\frac{F_x}{|\vec{F}_2|}\right) = \cos^{-1}\left(\frac{-212.13}{700}\right) = 107.64^\circ$$

$$\beta' = \cos^{-1}\left(\frac{F_y}{|\vec{F}_2|}\right) = \cos^{-1}\left(\frac{650}{700}\right) = 21.79^\circ$$

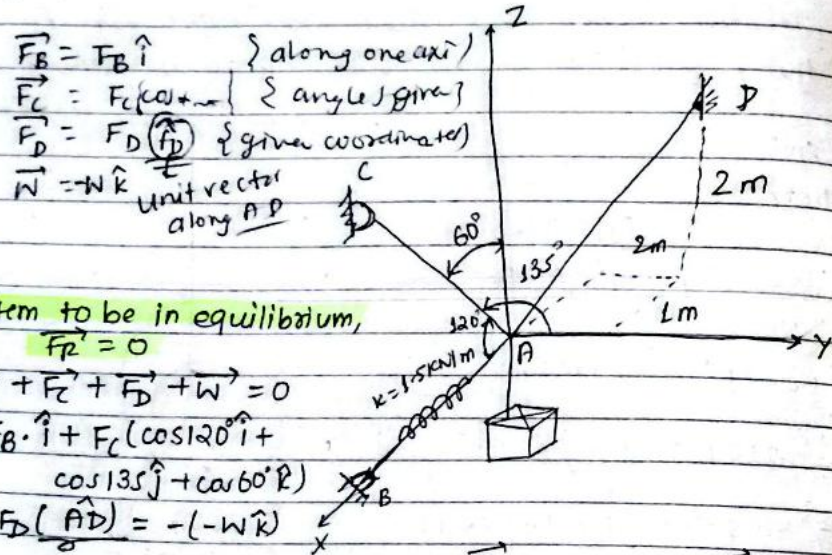
$$\gamma' = \cos^{-1}\left(\frac{F_z}{|\vec{F}_2|}\right) = \cos^{-1}\left(\frac{150}{700}\right) = 77.63^\circ$$



$$\left. \begin{array}{l} \sum \vec{F} = 0 \\ \sum \vec{M} = 0 \end{array} \right\} \text{vectorially}$$

$$\left. \begin{array}{l} F_C = 81.3 \\ F_D = 86.2 \\ F_B = 80.7 \end{array} \right\}$$

Q.15 Determine the tension in each cord use to support 100kg crate as shown.



For system to be in equilibrium,
 $\vec{F}_R = 0$

$$\vec{F}_B + \vec{F}_C + \vec{F}_D + \vec{W} = 0$$

$$\text{or, } F_B \hat{i} + F_C (\cos 120^\circ \hat{i} + \cos 135^\circ \hat{j} + \cos 60^\circ \hat{k}) + F_D (\hat{i}) = -(-W \hat{k})$$

$$\text{or, } F_B \hat{i} + -0.5 F_C \hat{i} - 0.707 F_C \hat{j} + 0.5 F_C \hat{k} + F_D \hat{i} + 2 F_D \hat{j} + 2 F_D \hat{k} = W \hat{k}$$

$$\text{or, } (F_B - 0.5 F_C + F_D) \hat{i} + (-0.707 F_C + 2 F_D) \hat{j} + (0.5 F_C + 2 F_D) \hat{k} = 981 \hat{k}$$

Equating coefficients,

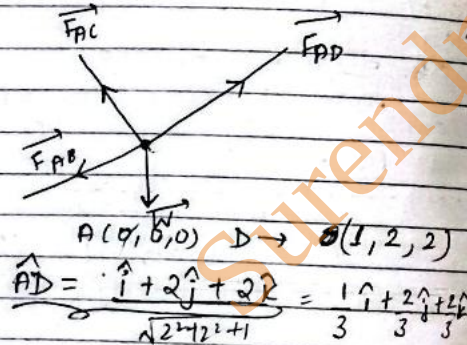
$$F_B - 0.5 F_C + F_D = 0$$

$$0.5 F_B - 0.707 F_C + 2 F_D = 0$$

$$0.5 F_B + 0.5 F_C + 2 F_D = 981$$

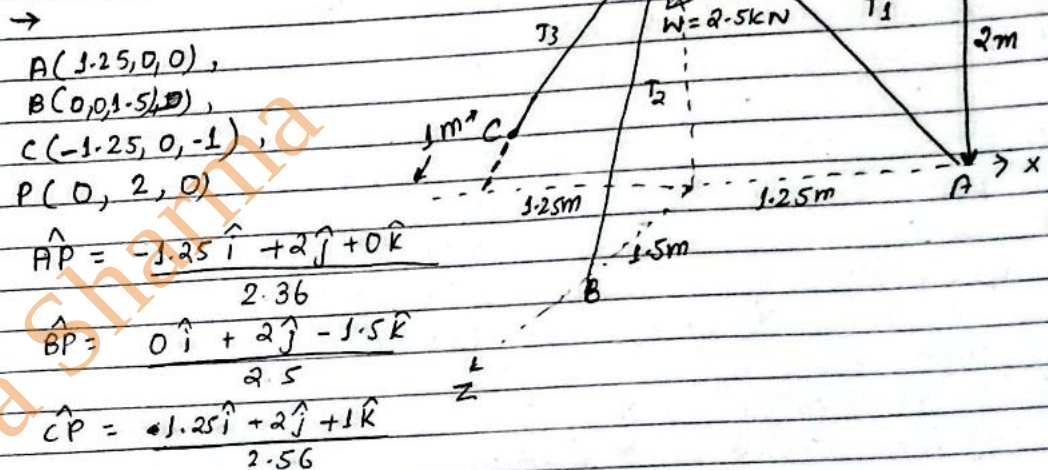
$$F_B = 119 \text{ N}$$

$$F_C = 812.76 \text{ N} \quad F_D = 287.81 \text{ N} \quad F_D = 861.93 \text{ N}$$



$$\left. \begin{array}{l} T_1 = 1.28 \text{ kN} \\ T_2 = 0.832 \text{ kN} \\ T_3 = 1.04 \text{ kN} \end{array} \right\}$$

Q.16 A tripod supports a load of 2.5 kN at point 'P' as shown. The end point A, B and C of the three legs lie in xz-plane. Determine force developed in each leg.



For above system to be in eqm.

$$\sum \vec{F} = 0$$

$$T_1 \vec{AP} + T_2 \vec{BP} + T_3 \vec{CP} = W$$

$$\text{or, } T_1 (-1.25 \hat{i} + 2 \hat{j}) + T_2 (2 \hat{j} - 1.5 \hat{k}) + T_3 (1.25 \hat{i} + 2 \hat{j} + 1 \hat{k}) = 2.5 \hat{j}$$

$$\text{or, } (-0.53 T_1 + 0.49 T_3) \hat{i} + (0.85 T_1 + 0.8 T_2 + 0.78 T_3) \hat{j} + (-0.6 T_2 + 0.39 T_3) \hat{k} = 2.5 \hat{j}$$

Equating coeff.

$$-0.53 T_1 + 0.49 T_3 = 0$$

$$0.85 T_1 + 0.8 T_2 + 0.78 T_3 = 2.5$$

$$0 T_1 - 0.6 T_2 + 0.39 T_3 = 0$$

Solving,

$$T_1 = 1.28 \text{ kN} \quad 1.1 \text{ kN}$$

$$T_2 = 0.832 \text{ kN} \quad 0.78 \text{ kN}$$

$$T_3 = 1.04 \text{ kN} \quad 1.2 \text{ kN}$$

Q.17 A force system shown

Couple & its Characteristics

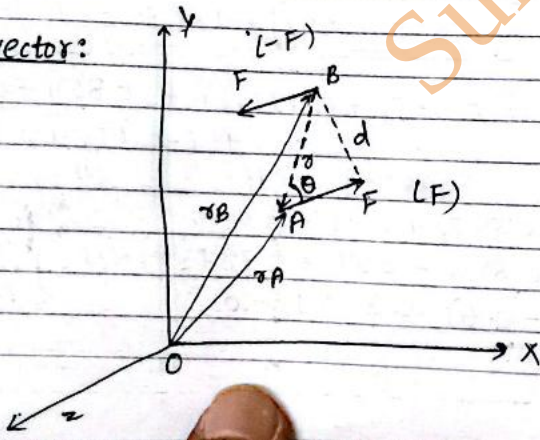
- Two equal and opposite forces acting at different point of different line of action is called couple.
- Two forces having same magnitude, parallel line of action & opposite sense is called couple.

* Characteristics of couple:-

1. The sum of component of two force in any direction is zero.
2. The sum of moments of two forces about a given point however is not zero.
3. While the two forces will not translate a body on which they act, they will tend to make it rotate.
4. The couple can only be balanced by a couple of opposite sense.
5. A number of coplanar couples can be reduced to simple couple whose magnitude will be equal to algebraic sum of moments of all couples.
6. Couple is a free vector. (→)

Proof:

Couple is a free vector:



F.21

$$\begin{aligned} \vec{M}_O &= \vec{r}_A \times \vec{F} + \vec{r}_B \times (-\vec{F}) \\ &= (\vec{r}_A - \vec{r}_B) \times \vec{F} \\ &= \vec{r} \times \vec{F} \end{aligned}$$

✓ $\vec{M}_O$ called moment of a couple is a vector perpendicular to the plane containing two forces & its magnitude is

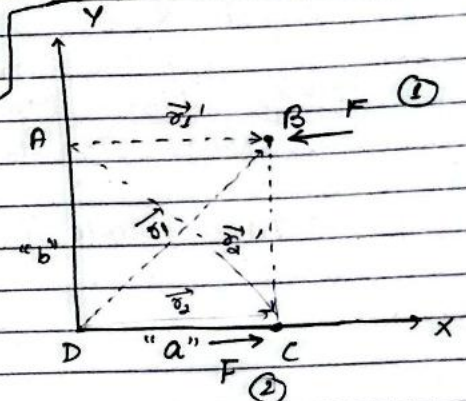
$$* \quad M_0 = r F \sin \theta = F d$$

where d is the $\perp$ distance between F & $(-F)$

✓ Sense of M_0 can be obtained from ~~left~~ ^{right} hand thumb rule.

* → Since the vector $\vec{r}$ is independent of choice of origin 'O' of the coordinate axes. The same moment ~~result~~ have been obtained if moments of F & $-F$ had been computed about different point 'O'.
Thus, moment 'M' of a couple is a free vector

Illustration: →



Consider rectangular plate acted upon by two equal & opposite forces at 'B' & 'C'.

$$\vec{M}_D = (a\hat{i} + b\hat{j}) \times (-F\hat{i}) = Fb\hat{k}$$

$$\begin{aligned} \vec{M}_A &= (b\hat{j}) \times (-F\hat{i}) + (-b\hat{j} + a\hat{i}) \times (F\hat{i}) \\ &= Fb\hat{k} \end{aligned}$$

$$\vec{M}_P = \vec{M}_D$$

Now,

$$\vec{M}_D = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2$$

$$= (a\hat{i} + b\hat{j}) \times (-F\hat{j}) + (a\hat{i}) \times (F\hat{j})$$

$$= -aF\hat{k} + 0 + aF\hat{k}$$

$$\vec{M}_A = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2$$

$$= (0\hat{i} + 0\hat{j}) \times (-F\hat{j}) + (-b\hat{j}) \times (F\hat{j})$$

Now

$$\vec{M}_D = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2$$

$$= (a\hat{i} + b\hat{j}) \times (-F\hat{j}) + (a\hat{i}) \times (F\hat{j})$$

$$= -aF\hat{k} + 0 + aF\hat{k}$$

$$= bF\hat{k}$$

$$\vec{M}_A = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2$$

$$= (0\hat{j}) \times (-F\hat{j}) + (a\hat{i} - b\hat{j}) \times (F\hat{j})$$

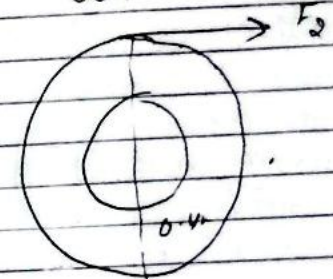
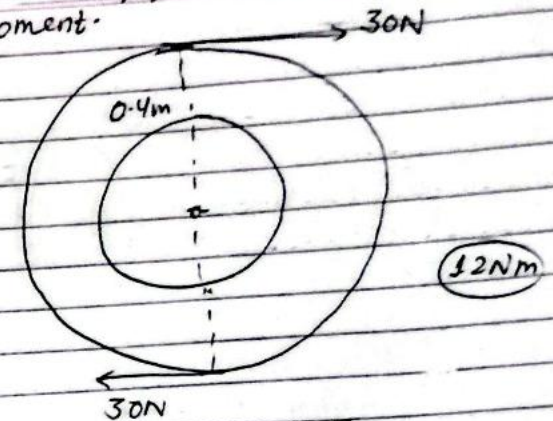
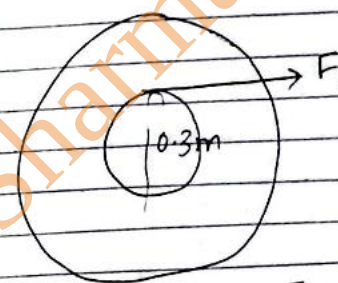
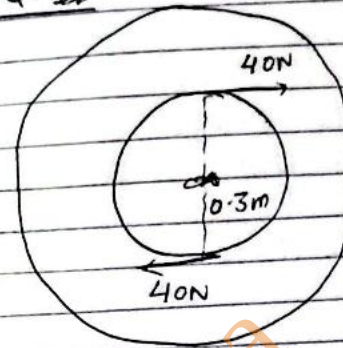
$$= 0 + aF\hat{k} - bF\hat{k}$$

$$= bF\hat{k}$$

Since, $\vec{M}_D = \vec{M}_A$ about two different points D & A.
 Magnitude & direction of couple is same.

Thus, couple is a free vector.

Q. 12 Illustrate with suitable example show diff. b/w couple & moment.



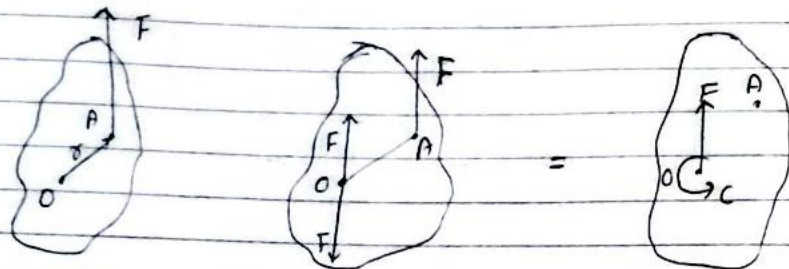
For same moment

$$F_1 = 80\text{N} = \frac{12 \times 2}{0.3}$$

$$F_2 = \frac{12 \times 2}{0.4} = 60\text{N}$$

} Greater force to be applied

Note: Replaced force = Resultant force of given system
 Couple = moment of given system of force about force replaced point
 # Resolution of a force into a force & a couple.



Condition: Force acting at A
 Requirement = Force at O

Procedure

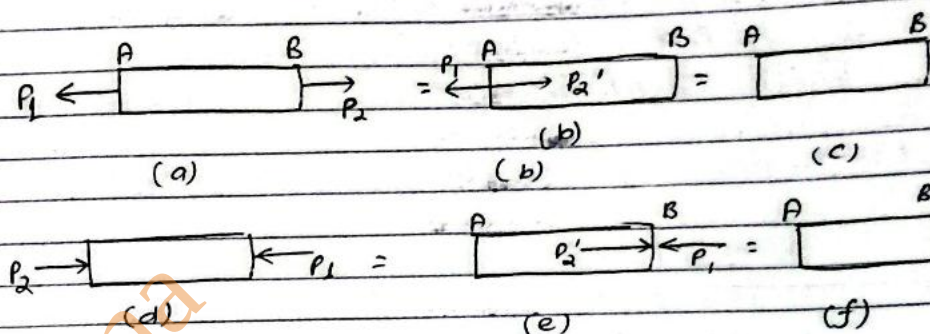
- (1) Introduce two equal and opposite Force F at O & (equilibrium of two forces)
- (2) Introduce couple from two forces at A & O couple represented by couple vector $\vec{C}$ (couple is a free vector)
- (3) Force couple system at O. (Equivalent system)

→ Thus, any force F acting on a rigid body can be moved to an arbitrary point 'O' provided that a couple is added whose moment is equal to moment of F about O.

→ The reverse of this theorem is also true

sliding vector.
 # Principle of transmissibility and its limitations:-

Limitations



- A short bar AB acted upon by equal and opposite forces (axial force) P_1 and P_2 [fig (a)]
- According to principle of transmissibility the force P_2 can be replaced by P_2' having same magnitude, direction & line of action but acting at A instead of B. [fig (b)]. The P_1 force P_1 & P_2' acting on the same body can be added and as these forces are equal and opposite their sum is equal to zero. Thus, in terms of external behaviour of the bar, the original system of forces shown in fig (a) is equivalent to no force at all i.e. fig (c).
- Similarly, force system in fig. (d) resolves in no force at all.
- From the point of mechanics of rigid body, the system shown in fig (a) and fig. (d) are thus equivalent. But, the internal forces and deformation produced by the two systems are clearly different.

* Principle of Transmissibility is applicable only for rigid body and when external effect is taken into account & internal effect is neglected

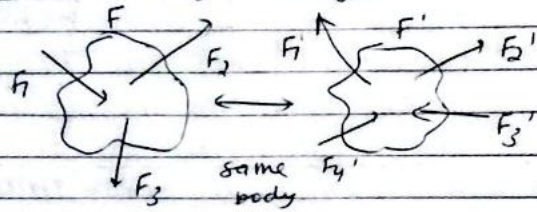
The bar in fig (a) is in tension and if not perfectly rigid, will increase in length slightly. Whereas, the bar in fig (d) is in compression and if not absolutely rigid, will decrease in length slightly.

Ques 1 Define equivalent and equipollent forces. (2)

→ Equivalent forces

(system of) Two forces are said to be equivalent if they have the same magnitude and direction i.e. they are equal and produce same moment about any point (i.e. same line of action).

For equivalent forces, system of forces are acting on same rigid body.



$$\sum \vec{F} = \sum \vec{F}' \quad \& \quad \sum \vec{M}_O = \sum \vec{M}_O'$$

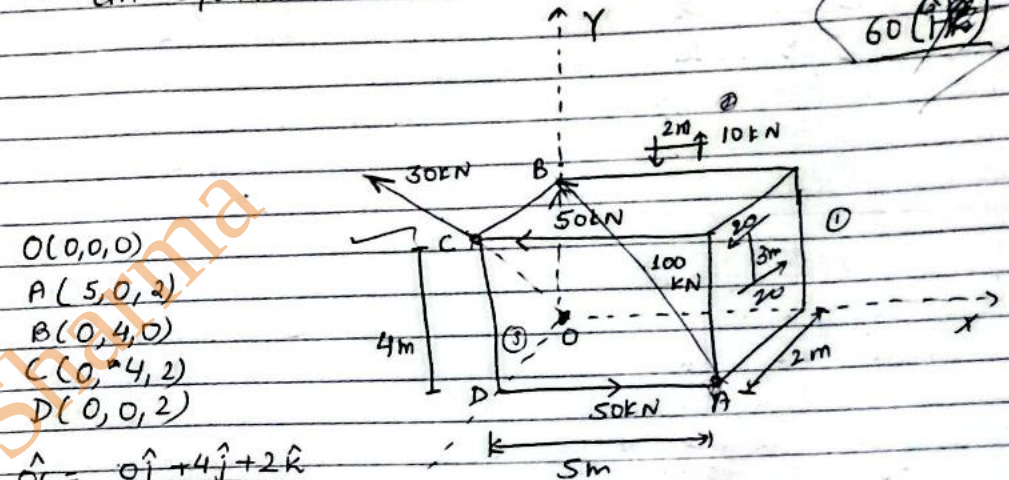
Equivalent force: Diff system of forces on a same rigid body.

Equipollent: Two force system are equipollent if the total force of the two systems are equal and also their total moment about an arbitrary point are same.

+60

(These two system are not equivalent)

Ques 2 Replace the force and couple system by an equivalent force & couple system at O.



$O(0,0,0)$
 $A(5,0,2)$
 $B(0,4,0)$
 $C(0,4,2)$
 $D(0,0,2)$

$$\hat{OC} = \frac{0\hat{i} + 4\hat{j} + 2\hat{k}}{2\sqrt{5}}$$

$$\hat{AB} = \frac{-5\hat{i} + 4\hat{j} - 2\hat{k}}{3\sqrt{5}}$$

→ Solⁿ:-

Here,

Resultant force ($\vec{F}$)

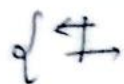
$$\begin{aligned} &= 30(\hat{OC}) + 100(\hat{AB}) \\ &= 30\left(\frac{4\hat{j} + 2\hat{k}}{2\sqrt{5}}\right) + 100\left(\frac{-5\hat{i} + 4\hat{j} - 2\hat{k}}{3\sqrt{5}}\right) \\ &= -74.5\hat{i} + 86.46\hat{j} - 16.84\hat{k} \end{aligned}$$

$$\begin{aligned} \vec{R} &= \vec{F}_1 + \vec{F}_2 \\ &= F_1 \hat{f}_1 + F_2 \hat{f}_2 \end{aligned}$$

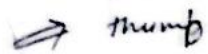
$$\begin{aligned} \vec{M}_O &= \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 \\ &= \vec{C}_1 + \vec{C}_2 + \vec{C}_3 \end{aligned}$$

$$\begin{aligned} \text{resultant moment } (\vec{M}_O) &= \vec{OC} \times \vec{F}_1 + \vec{OB} \times \vec{F}_2 + 60\hat{i} \\ &= +200\hat{k} + 20\hat{k} \end{aligned}$$

$$\begin{aligned} &= (4\hat{j} + 2\hat{k}) \times \left\{ 30\left(\frac{4\hat{j} + 2\hat{k}}{2\sqrt{5}}\right) \right\} + (4\hat{j}) \times \left\{ 100\left(\frac{-5\hat{i} + 4\hat{j} - 2\hat{k}}{3\sqrt{5}}\right) \right\} \\ &= 60\hat{i} + 220\hat{k} \end{aligned}$$



curl of
finger



couple
representation

$$= 14.9 \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 4 & 0 \\ -5 & 4 & -2 \end{vmatrix} + 60\hat{i} + 220\hat{k}$$

$$= 14.9 \{ -8\hat{j} + 20\hat{k} \} + 60\hat{i} + 220\hat{k}$$

$$= -59.2\hat{j} + 518\hat{k}$$

Thus,

Resultant force on a system is $\vec{F}_R = -74.5\hat{i} + 86.46\hat{j} - 164\hat{k}$

Resultant couple/moment on a system about point O $\vec{M}_O = -59.2\hat{j} + 518\hat{k}$



Wrench

Resultant of force & moment for a system of force:-

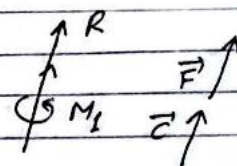
- Simplest resultant of general force system
- Simplest resultant of coplanar force system
- Simplest resultant of parallel force system

Eg: Tightening & opening of the screw by a screw driver are example of positive and negative wrench respectively

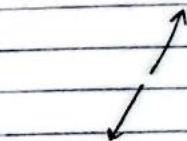
Wrench

A wrench consist of a force and a couple with same line of action.

If the directions are same, it is called positive wrench. & If direction are opposite, it is called negative wrench.



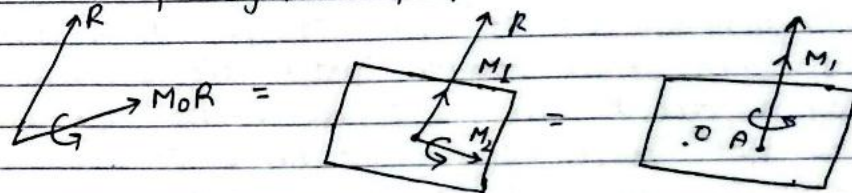
+ve wrench



-ve wrench

*X--

Reduction of system of force to wrench



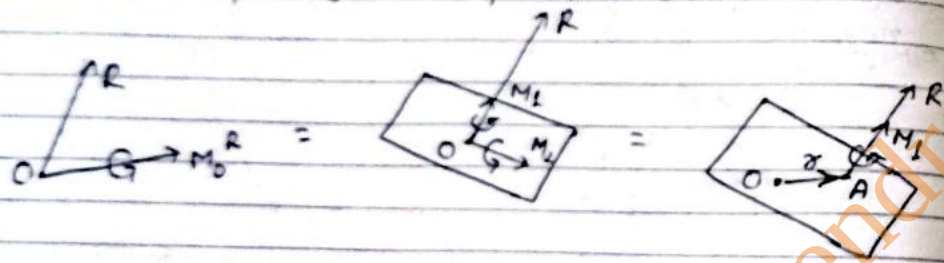
* The action of wrench is physically equivalent to push or pull along an axis plus twist about that axis.

→ The line of action of force is axis of wrench. The ratio of couple by force is called pitch of wrench.

→ Thus, wrench consist of two collinear vector, force vector & couple vector.

$$\text{Pitch}(P) = \frac{\text{couple}(C)}{\text{Force}(F)}$$

* Reduction of system of force to Wrench:-



In general case of system of force in space, the equivalent force couple system at O consists of a force R and a couple vector $M_o R$ which are not perpendicular and neither of which is zero. Thus, the system of force cannot be reduced to a single force or a single couple.

The couple vector however may be replaced by two other couple vectors obtained by resolving $M_o R$ into components M_1 along R and components M_2 in a plane $\perp$ to R . The couple vector M_2 and R may then be replaced by a single force R acting along a new line of action. The original system of force thus reduces to R and couple vector M_1 i.e. R and couple acting in the plane perpendicular to R . This particular force-couple system is called wrench.

$$M_1 = PR$$

$$P = \frac{M_1}{R} = \frac{R \cdot M_o R}{R^2}$$

$$M_1 + \vec{r} \times R = M_o R$$



Q. Two forces of same magnitude P acts on a side of an cube of side ' a '. as shown. Replace the two forces by an equivalent wrench & Determine

- Resultant force and pitch of the wrench
- The point where the axis of wrench intersects y - z plane

→ Solⁿ:-

$$\vec{r}_1 = a\hat{i} + a\hat{j} + 0\hat{k}$$

$$\vec{r}_2 = 0\hat{i} + a\hat{j} + a\hat{k}$$

$$\vec{R} = \vec{F}_1 + \vec{F}_2$$

$$= P\hat{i} + P\hat{j}$$

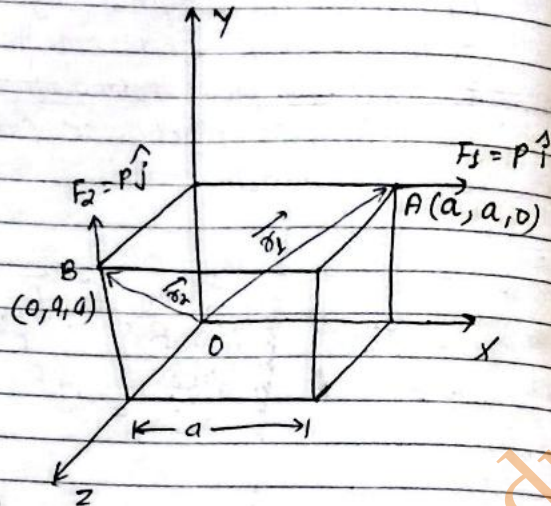
$$\vec{M}_O^R = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2$$

$$= (a\hat{i} + a\hat{j}) \times (P\hat{i})$$

$$+ (a\hat{j} + a\hat{k}) \times (P\hat{j})$$

$$= -aP\hat{k} - aP\hat{i}$$

$$= -aP\hat{i} - aP\hat{k}$$



* Resultant force, R (magnitude & θ)

$$R = \sqrt{P^2 + P^2 + 0^2} = \sqrt{2} P$$

$$\theta = \tan^{-1} \left(\frac{P}{\sqrt{2}P} \right) > 35.26^\circ$$

$$\theta = \tan^{-1} \left(\frac{P}{P} \right) = 45^\circ \text{ (with } m\text{-axis)}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ 1 & 1 & 0 \end{vmatrix}$$

$$* \text{ Pitch of wrench } (P) = \frac{\vec{R} \cdot \vec{M}_O^R}{R^2}$$

$$= \frac{-aP(\hat{i} + \hat{j}) \cdot (\hat{i} + \hat{k})}{(\sqrt{2}P)^2}$$

$$= \frac{-aP^2}{2P^2}$$

$$= -\frac{a}{2}$$

$$* \text{ Axis of wrench}$$

$$\vec{M}_1 = P\vec{R} = \left(-\frac{a}{2}\right)(P\hat{i} + P\hat{j})$$

$$= -\frac{aP}{2}(\hat{i} + \hat{j})$$

* To find point where wrench intersects the yz plane
Moment of wrench about O is equal to moment resultant $\vec{M}_O^R$ of original system

$$\vec{M}_1 + \vec{r} \times \vec{R} = \vec{M}_O^R$$

$$\Rightarrow \vec{M}_1 + (x\hat{i} + y\hat{j} + z\hat{k}) \times (P\hat{i} + P\hat{j}) = -aP(\hat{i} + \hat{k})$$

$$\Rightarrow -\frac{aP}{2}(\hat{i} + \hat{j}) - aP(\hat{i} - \hat{k}) \times (\hat{i} + \hat{j}) = -aP(\hat{i} + \hat{k})$$

For y - z plane, $x=0$

$$\Rightarrow -\frac{aP}{2}(\hat{i} + \hat{j}) + P(x\hat{i} + y\hat{j} + z\hat{k}) \times (\hat{i} + \hat{j}) = -aP(\hat{i} + \hat{k})$$

$$\Rightarrow -\frac{aP}{2}(\hat{i} + \hat{j}) + \hat{i}(-Pz) - Pz\hat{j} + (Pa - Py)\hat{k} = -aP(\hat{i} + \hat{k})$$

Equating coeff.

$$\frac{-ap}{2} - pz = -ap$$

$$z = \frac{-ap}{2} - \frac{q}{2} + a = \frac{a}{2}$$

$$\frac{-ap}{2} - apy = -ap$$

$$y = a$$

Q19 replace the two wrenches by an equivalent wrench.

→ Solⁿ:

Resultant forces ($\vec{R}$)

$$= \vec{F}_1 + \vec{F}_2$$

$$= 50\hat{i} + 40\hat{j}$$

$$\vec{M}_O \vec{R} = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2$$

$$+ \vec{C}_1 + \vec{C}_2$$

$$= 0 + (6\hat{k}) \times (50\hat{i})$$

$$+ (-100\hat{j}) + (80\hat{i})$$

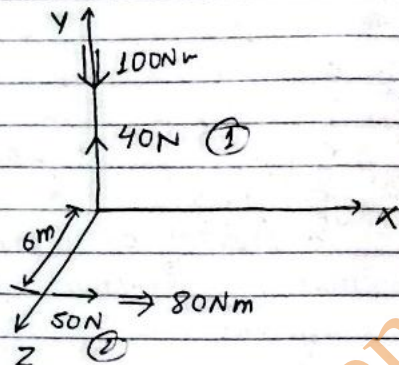
$$= 300\hat{j} - 100\hat{j} + 80\hat{i}$$

$$= 80\hat{i} + 200\hat{j}$$

$$R = \sqrt{50^2 + 40^2} = 64.03 \text{ N}$$

$$p = \frac{\vec{R} \cdot \vec{M}_O \vec{R}}{R^2} = \frac{(50\hat{i} + 40\hat{j}) \cdot (80\hat{i} + 200\hat{j})}{4100}$$

$$p = \frac{120}{41}$$



146.

↑ $\hat{i}$ $\hat{j}$ $\hat{k}$ Theory
 $\hat{i}$ $\hat{j}$ $\hat{k}$ balls & cables
 $\hat{i}$ $\hat{j}$ $\hat{k}$ force & couple?

$$\vec{M}_1 = p \vec{R} = \frac{120}{41} (50\hat{i} + 40\hat{j})$$

$$= 146.34\hat{i} + 117.07\hat{j}$$

Now,

Let, (x, y, z) be point of application of wrench

$$\vec{r} \times \vec{R} + \vec{M}_1 = \vec{r} \times \vec{M}_1$$

$$\text{or, } (x\hat{i} + y\hat{j} + z\hat{k}) \times \frac{120}{41} (50\hat{i} + 40\hat{j}) = 80\hat{i} + 200\hat{j}$$

$$\text{or, } \frac{120}{41} \{ 40z\hat{i} + 50z\hat{j} + (40x - 50y)\hat{k} \} = 80\hat{i} + 200\hat{j}$$

$$\text{Equating coeff.}$$

$$-40z \times \frac{120}{41} = 80$$

$$\text{equating coeff.}$$

$$50z = 82.93$$

$$\Rightarrow z = 1.65$$

$$\text{or, } (x\hat{i} + y\hat{j} + z\hat{k}) \times (50\hat{i} + 40\hat{j}) = 80\hat{i} + 200\hat{j}$$

$$+ \frac{120}{41} (50\hat{i} + 40\hat{j})$$

$$= 80\hat{i} + 200\hat{j}$$

Q20 Compute a wrench out of a force

$$\vec{F} = 5\hat{i} + 7\hat{j} + 18\hat{k} \text{ acting at } (4, 4, 5)$$

and a couple $\vec{C} = 18\hat{i} + 7\hat{j} + 5\hat{k}$. Indicate line of action in xy-plane.

→ Solⁿ:

$$\vec{R} = \vec{F} = 5\hat{i} + 7\hat{j} + 18\hat{k}$$

$$\vec{M}_1 = 2.878\hat{i} + 4.0027\hat{j} + 10.357\hat{k}$$

$$(x, y, z) = (2.44, 2.89, 0)$$

$$\vec{M}_O \vec{R} = \vec{r} \times \vec{F} + \vec{C}$$

$$= (4\hat{i} + 4\hat{j} + 5\hat{k}) \times (5\hat{i} + 7\hat{j} + 18\hat{k}) + (18\hat{i} + 7\hat{j} + 5\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 5 \\ 5 & 7 & 18 \end{vmatrix} + (18\hat{i} + 7\hat{j} + 5\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 5 \\ 5 & 7 & 18 \end{vmatrix} + (18\hat{i} + 7\hat{j} + 5\hat{k})$$

$$= 37\hat{i} - 47\hat{j} + 8\hat{k} + 18\hat{i} + 7\hat{j} + 5\hat{k}$$

$$= 55\hat{i} - 40\hat{j} + 13\hat{k}$$

$$R = \sqrt{5^2 + 7^2 + 18^2} = 19.94$$

$$p = \frac{\vec{R} \cdot \vec{M}_O^R}{R^2} = \frac{(5\hat{i} + 7\hat{j} + 18\hat{k}) \cdot (55\hat{i} - 40\hat{j} + 13\hat{k})}{398}$$

$$p = \frac{229}{398}$$

$$\vec{M}_1 = p\vec{R} = \frac{229}{398} (5\hat{i} + 7\hat{j} + 18\hat{k})$$

$$\vec{M}_1 = 2.876\hat{i} + 4.027\hat{j} + 10.357\hat{k}$$

Let, (x, y, z) be point on XY-plane through which wrench passes.

for XY-plane $z = 0$

Now,

From principle of moment,

$$\vec{r} \times \vec{R} + \vec{M}_1 = \vec{M}_O^R$$

$$\text{or, } (x\hat{i} + y\hat{j}) \times (5\hat{i} + 7\hat{j} + 18\hat{k}) + 2.876\hat{i} + 4.027\hat{j} + 10.357\hat{k}$$

$$\text{or, } \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & 0 \\ 5 & 7 & 18 \end{vmatrix} + 2.876\hat{i} + 4.027\hat{j} + 10.357\hat{k} = 55\hat{i} - 40\hat{j} + 13\hat{k}$$

$$\text{or, } 18y\hat{i} - 18x\hat{j} + (7x - 5y)\hat{k} = 52.214\hat{i} - 44.027\hat{j} + 2.643\hat{k}$$

Equating coeff.

$$\begin{cases} 18y = 52.214 \\ -18x = -44.027 \end{cases} \Rightarrow \begin{cases} y = 2.9 \\ x = 2.45 \end{cases}$$

$\therefore$ The point on y-z plane is $(2.45, 2.9, 0)$

$$108P + 166.15Q - C = 74976000$$

$$C = 216000 \text{ Nmm}$$

$$P = 1000 \text{ N}, Q = 1300 \text{ N}$$

Q.21 Four system shown consist of couple 'C' and four forces. If the resultant of this system is 75000 Nmm counter clockwise couple of force

Determine P, Q & C.

$\rightarrow$ Solⁿ:-

Let us calculate the moment given by each forces about point O.

so, From principle of moment and question,

$$\vec{M}_R = +75000 \text{ } \{ \text{counter clockwise} \}$$

so,

$$\vec{M}_{QO} + \vec{M}_{PO} + \vec{C} + \vec{M}_{F'O} = \vec{M}_R$$

$$\text{or, } (180\hat{j}) \times (-Q \cos(22.62^\circ)\hat{j})$$

$$+ (180\hat{j}) \times (P \sin(36.87^\circ)\hat{j})$$

$$- C + (60\hat{j}) \times (400\hat{i}) = 75000\hat{k}$$

$$\text{or, } 166.15Q\hat{k} + 108P\hat{k} - C\hat{k} - 24000\hat{k} = 75000\hat{k}$$

on,

Equating coeff.

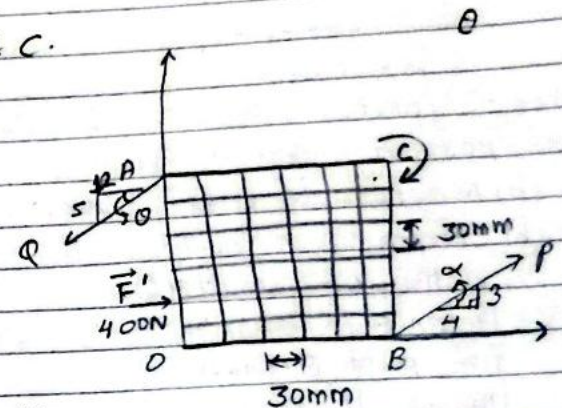
$$108P + 166.15Q - C = 99000 \quad \text{--- (i)}$$

$$P \cos(36.87^\circ) - Q \cos(22.62^\circ) = 400 \quad \text{--- (ii)}$$

$$P \sin(36.87^\circ) - Q \sin(22.62^\circ) = 0 \quad \text{--- (iii)}$$

Solving,

$$P = 625 \text{ N}, Q = 975 \text{ N}, C = 130497 \text{ Nm}$$



each box $\rightarrow (30 \times 30) \text{ mm}^2$

$$\theta = \tan^{-1}\left(\frac{5}{2}\right) = 22.62^\circ$$

$$\alpha = \tan^{-1}\left(\frac{3}{4}\right) = 36.87^\circ$$

$$\{ y \times \Sigma F_x + x \times \Sigma F_y = M_0 \}$$

Q22 Three forces and couple are applied to a bracket as shown. Find

- (a) Magnitude and direction of resultant
 (b) Perpendicular distance from point O to line of action of resultant.
 (c) Distance in x-axis from point O where the resultant cuts.

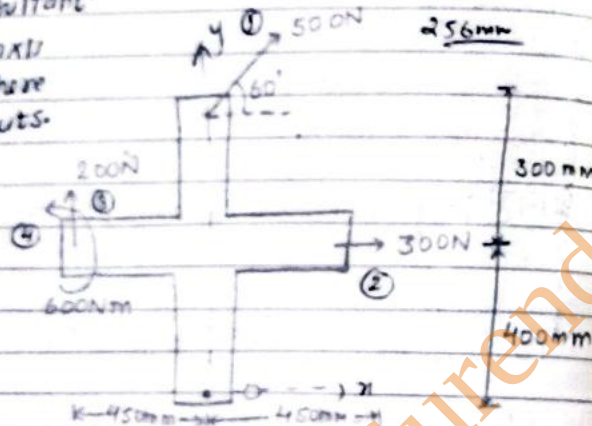
→ Solution:

$$\textcircled{a} \Sigma F_x = 300 + 250 = 550 \text{ N}$$

$$\Sigma F_y = 633.01 \text{ N}$$

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2} = \sqrt{(550)^2 + (633.01)^2} = 838.6 \text{ N}$$

$$\theta = \tan^{-1} \left(\frac{\Sigma F_y}{\Sigma F_x} \right) = \tan^{-1} \left(\frac{633.01}{550} \right) = 49^\circ$$



$$\{ \cdot \uparrow F \} \rightarrow \text{Moment outside} \quad \{ F \uparrow \cdot O \} \rightarrow \text{Moment inside} \quad \{ \cdot \uparrow F \} \rightarrow \text{Anticlockwise} \quad \{ \cdot \uparrow F \} \rightarrow \text{Clockwise}$$

$$(b) F \cdot d = M_{O1} + M_{O2} + M_{O3} + C$$

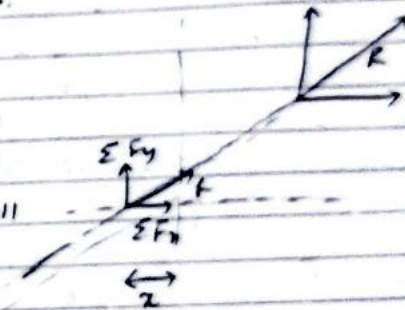
$$838.6 \times d = -175000 + 120000 + 90000 + 600000$$

$$\text{or } 838.6 \times d = -500 \cos(60^\circ) \times 700 + 300 \times 400 + 200 \times 450 + 600000$$

$$\text{or } d = \frac{-175000 - 120000 - 90000 + 600000}{838.6} = 256.38 \text{ mm}$$

$$d = 256.38 \text{ mm}$$

(c) At the point when resultant cuts x-axis, y-coordinate is 0 at that point and only ΣF_y will be producing moment.



So, From Varignon's theorem.

$$y \times \Sigma F_x + x \times \Sigma F_y = M_0$$

$$\text{or } x \times 633.01 = 838.6 \times 256.38$$

$$\therefore x = 339.64 \text{ mm}$$

(OR) Let, (x, y) be the point,

$$\vec{r} \times \vec{F} = \vec{M}_1 + \vec{M}_2 + \vec{M}_3 + \vec{C}$$

$$\text{or } (x\hat{i} + y\hat{j}) \times (550\hat{i} + 633.01\hat{j}) = \{ -500 \cos(60^\circ) (\hat{i} \times \hat{j}) + 700\hat{j} \} + \{ -300\hat{i} \times (400\hat{j}) \} + \{ 200\hat{j} \times (-450\hat{i}) \} + 600000 \hat{k}$$

$$\vec{r} \times \vec{F} = \vec{F} \times \vec{r} = 0$$

$$\text{or, } 633.01 \hat{k} = -175000 \hat{k} - 120000 \hat{k} + 90000 \hat{k} + 600000 \hat{k}$$

$$\text{or, } 633.01 \hat{k} = 215000 \hat{k}$$

$$\text{Thus, } x = \underline{\underline{339.64 \text{ mm}}}$$

Beam & Frame

Structure

What is structure?

- Assembly of different element to make a system to obtain a stable system
- In structure, combination of parts function as a single system.
- Engineers are problem solver. They utilize the available resources for the mankind.

Structural Engineering

- Engineering which deals with the analysis and design of structure.

<u>Analysis</u>	<u>Design</u>
↓ To determine <u>Response</u> of the structure, ↓ Reaction, Deflection, <u>Internal force, etc.</u> (Axial, shear, Bending Moment)	↓ To determine the dimensional properties (or size), material properties, quantity & quality

Requirements of well-design structure:

- (i) Strong, stable and safe enough to carry load.
- (ii) Satisfactory service.
- (iii) Economical ———— proper design
use of local material
- (iv) Aesthetic consideration

↪ Satisfactory service → No excessive deformation, &
 ↳ No discomfort ↓
 ↳ According to purpose of that structure deflection/vibration/
 settlement (or) etc.

Technically,

3s → Strength
→ Stiffness
→ stability

with due consideration of economy & aesthetic

① Strength: Capacity of the structure or its elements to resist applied load without any collapse or failure.

② Stiffness: Capacity of the structure or its elements to resist excessive deformation due to applied load.

Core Technically,

Force required for unit displacement is stiffness.

③ Stability: Ability of the structure or its elements to retain in its original position due to application of external load.

Types of load:

- (1) Dead load, (static load or self weight)
- (2) Live load,
- (3) Seismic load,
- (4) Hydrostatic load,
- (5) Soil Pressure load, (retaining walls)
- (6) Wind load, etc.

* Dead load: static load & self weight
↳ ^{wall} partition load, Floor finishes

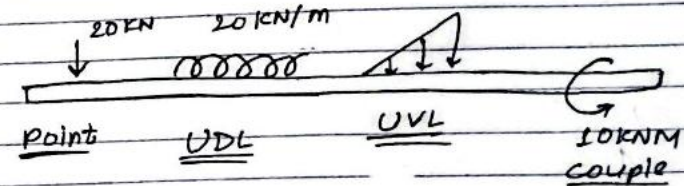
* Live load: Purpose (उद्देश्य को लागू बनाएँ को लोड) ^{उपयोग को} User.

In classroom, students are live load

In Bridges, vehicles are live load

All these loads are analysed under following load cases:

- (i) Point load
- (ii) Uniformly Distributed Load (UDL)
- (iii) Uniformly varying load (UVL)
- (iv) Couple.



Types of support :-

- (1) Roller support : resist in one direction
- (2) Hinge support : resist in two $\perp$ er direction
- (3) Fixed support
- (4) Linked support (Tension in cables)

} resist in translational motion

→ Remove support & add reaction.

(1) Roller support

→ symbol  or 

→ resist in one direction

→ resist in translational direction

→ It gives rise to 1 force reaction

(2) Hinge support

→ Resist in two direction

→ Symbol: 

→ gives rise to 2 force reaction in direction.

(3) Fixed support

→ 

→ resist 2 force and 1 couple.

→ give rise to 2 force reaction & 1 moment

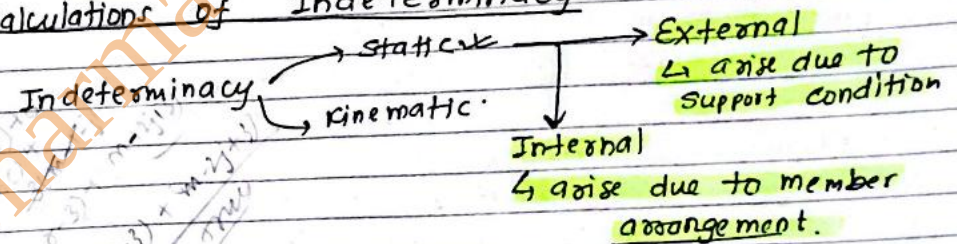
(4) Link support

↳ Tension in cables, rope

⊙ Determinate and Indeterminate structure
If the unknown reactions can be determined using equations of static equilibrium then such structure is called determinate structure.

And,
If the unknown reactions can't be determined using equations of static equilibrium then such structure is called indeterminate structure.

⊙ Calculations of Indeterminacy :-



Formula :-

	Beam	Frame	Trauss
Total =	$r - (3 + c)$	$(3m + 0) - (3j + c)$	$(m + r - 2j)$
External =	$r - (3 + c)$	$r - (3 + c)$	$(r - 3)$
Internal =	0	$\frac{3 \times n_l}{3(m - j + 1)}$	$(m - 2j + 3)$

→ due to reaction (support & internal hinge)

where, r = number of reactions

c = equation ^{due to} conditions

j = number of joints

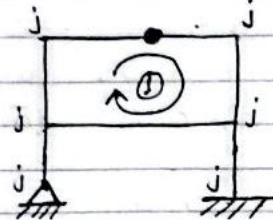
m = number of members

n_l = number of closed loops



Joint (nodes)

$$\begin{aligned} m &= 7 \\ j &= 7 \\ c &= 1 \\ r &= 5 \end{aligned}$$



(Frame)

Q. Determine number of indeterminacy.

→ For frame

$$m = 6$$

$$c = 1$$

$$r = 5$$

$$j = 6$$

$$TSI = (3m + r) - (3j + c) = 4$$

$$\text{External SI} = r - (3 + c) = 5 - (3 + 1) = 1$$

$$\text{Internal SI} = 3 \times n_h = 3 \times 1 = 3$$

① Note: Total Static Indeterminacy = External + Internal SI

Equation of conditions:-

Sometimes, indeterminate structure can be made determinate by providing internal hinge (by construction). say at point C. Then,

$$(\sum M_C)_{\text{either left or right}} = 0$$



Internal hinge → denoted by Dark Black Dot then, $c = 1$

One Internal hinge → 1 extra equation.

4 unknown, 3 eqs

Indeterminate

4 unknown, 4 eqs

Determinate

For frame,

If

$TSI = (3m + r) - (3j + c) = 0$, the structure is statically determinate and stable

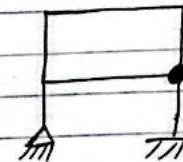
> 0 , indeterminate

< 0 , unstable & it's just a mechanism

Internal hinge (•) can be considered as joint as well.

$$c = \sum m - 1 \quad \left\{ \begin{array}{l} m \rightarrow \text{number of members connected to hinge} \end{array} \right.$$

eg:



$$c = 3 - 1 = 2$$

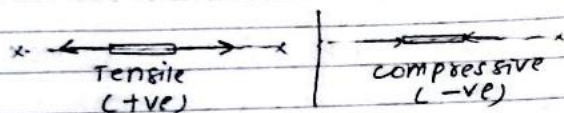
* Don't get confused that horizontal axis as longitudinal & vertical axis as transverse axis
 * It might not be always (True for Beam only)

Internal force in a structural member

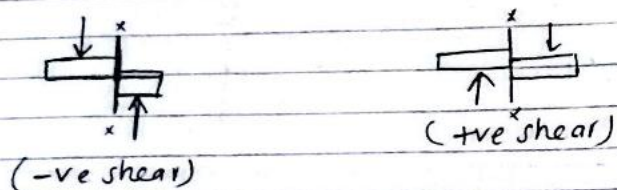
- (i) Axial force $\rightarrow$ tensile (+ve) compressive (-ve)
- (ii) Shear force $\rightarrow$ transverse force
- (iii) Bending moment $\rightarrow$ sagging (+ve) hogging (-ve)
- (iv) Torsion (Not now) $\rightarrow$ Twist about longitudinal axis

(AF) -

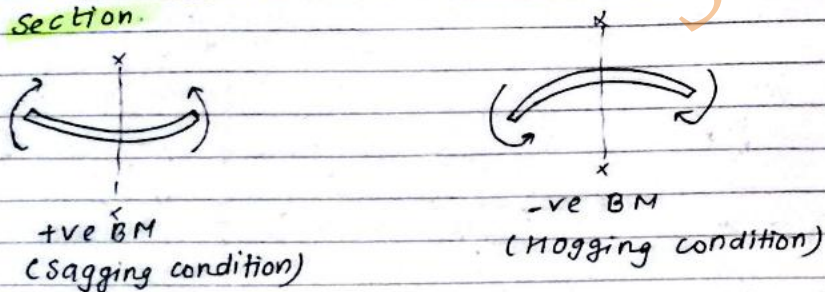
- (i) Axial Force: At any cross-section of member, it is the algebraic sum of all forces acting parallel to the longitudinal axis on either side of the section. A tensile is +ve and compressive is negative



- (ii) Shear Force (SF): It is the algebraic sum of all forces acting transverse to the member on either side of the section.



- (iii) Bending moment (BM): It is the algebraic sum of all the moments of all the forces on either side of the section.



In Beam Analysis
 Left to right movement $\rightarrow$ +ve
 Right to left movement $\rightarrow$ -ve
 Sign convention

Relationship b/w rate of loading, bending moment and shear force:

$\rightarrow$ Consider a beam, as shown. Take a elementary strip of length Δx .

$$\rightarrow \Sigma F_y = 0 \quad (\uparrow +ve)$$

$$V - (w \Delta x) - (V + \Delta V) = 0$$

$$\frac{\Delta V}{\Delta x} = -w$$

taking limit,

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta V}{\Delta x} = \frac{dV}{dx} = -w$$

$$\therefore \frac{dV}{dx} = -w$$

$$\rightarrow \Sigma M_B = 0$$

$$(M + \Delta M) - (M + V \Delta x) + w \Delta x \cdot \frac{\Delta x}{2} = 0$$

neglecting small quantities,

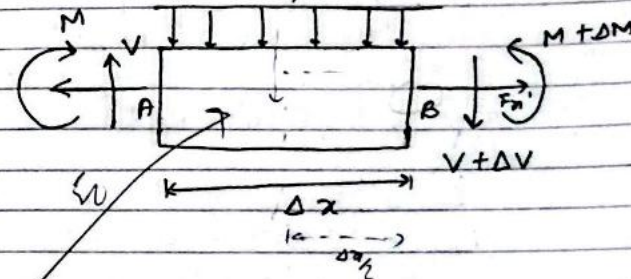
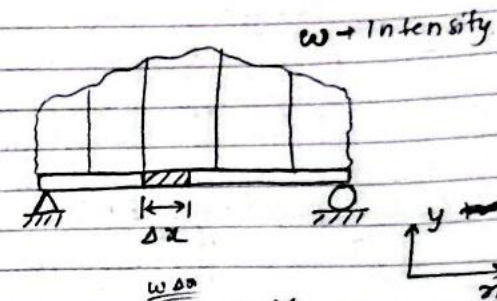
$$\frac{\Delta M}{\Delta x} = V$$

Taking limit, $\Delta x \rightarrow 0$

$$\frac{dM}{dx} = V$$

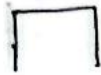
$\rightarrow$ Relationship b/w bending moment & shear force.

$\int dM = \int V dx \Rightarrow M_2 - M_1 = \int V dx \Rightarrow$ change in Bending moment b/w 2 points = Area under shear curve b/w those two points.



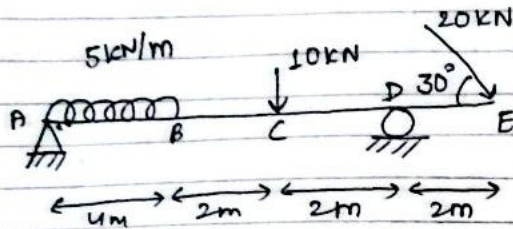
$\int dM = \int V dx$
 $M_2 - M_1 = \int V dx$
 change in moment = Area under shear curve

Portal Frame
(1 Bay 1 storey)

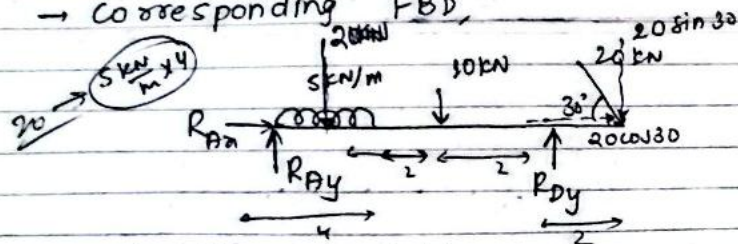


Analyze the beam

Q.1 Analyze the beam as shown.



→ Corresponding FBD



Equations of static eqm

(i) $\sum F_x = 0$ ($\rightarrow +ve$)

$$\sum F_x = 0$$

$$R_{Ax} + 20 \cos 30 = 0$$

$$R_{Ax} = -17.32 \text{ N}$$

(ii) $\sum M_D = 0$ ($\uparrow +ve$)

$$-R_{Ay} \times 8 + 20 \times 6 + 10 \times 2 - (20 \sin 30) \times 2 = 0$$

$$\therefore R_{Ay} = 15 \text{ kN}$$

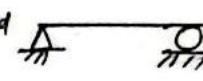
(iii) $\sum F_y = 0$ ($\downarrow +ve$)

$$-R_{Ay} + 20 + 10 - R_{Dy} + 20 \sin 30 = 0$$

$$\therefore R_{Dy} = 25 \text{ kN}$$

Types of Beam

(1) Simply supported Beam

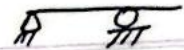


(2) Cantilever

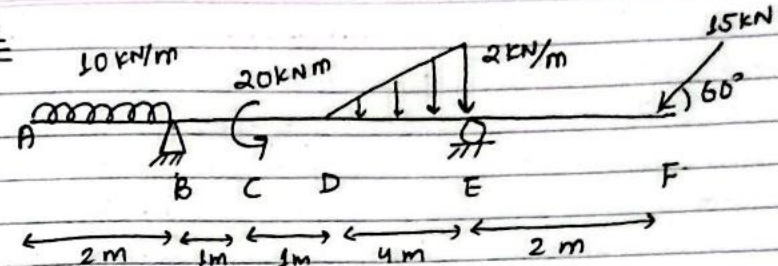
(3) Propped Cantilever



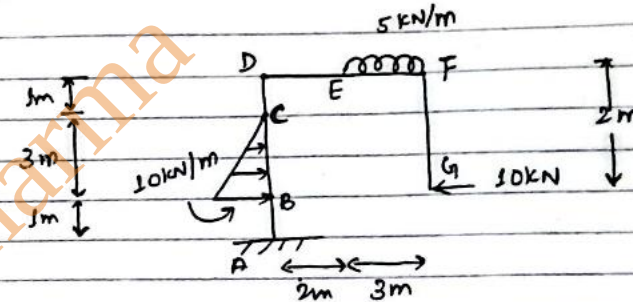
(4) Overhanging



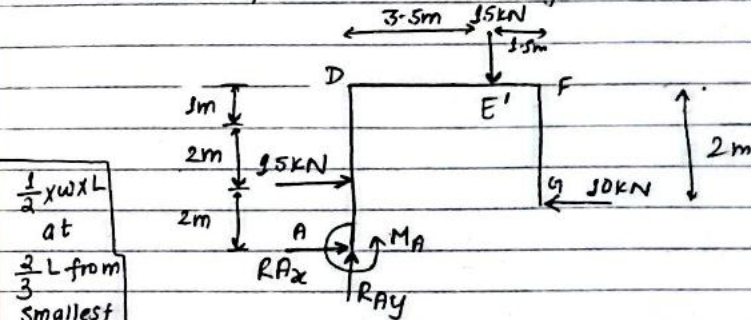
Q.2



Q.3



→ FBD of above beam system is-



$\frac{1}{2} \times w \times L$
at
 $\frac{2}{3} L$ from
Smallest
Force

→ For above system to be in static equilibrium,

(i) $\sum F_x = 0$ ($\rightarrow +ve$)

$$R_{Ax} + 15 - 10 = 0$$

$$\therefore R_{Ax} = -5 \text{ kN}$$

$$(ii) \sum F_y = 0 \quad (\downarrow +ve)$$

$$-R_{Ay} + 15 = 0$$

$$\Rightarrow R_{Ay} = 15 \text{ kN}$$

(iii) Considering E' as moment center,

$$\sum M_{E'} = 0 \quad (\curvearrowright +ve)$$

$$0 \Rightarrow 15 \times 0 + 15 \times 3 - 10 \times 2 + M_A = 0 + R_{Bx} \times 5$$

$$0 \Rightarrow M_A = -25 \text{ kNm} \quad \text{ie. clockwise moment}$$

$$0 \Rightarrow 25 - 25 - 15 \times 3.5 + M_B = 0$$

$$M_B = 52.5 \text{ kNm}$$

Anticlockwise moment

Imp

$$\frac{dv}{dx} = -w$$

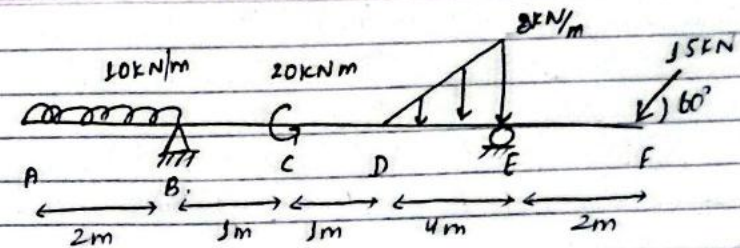
$$0 \Rightarrow dv = -w dx$$

$$\text{or, } \int dv = \int -w dx$$

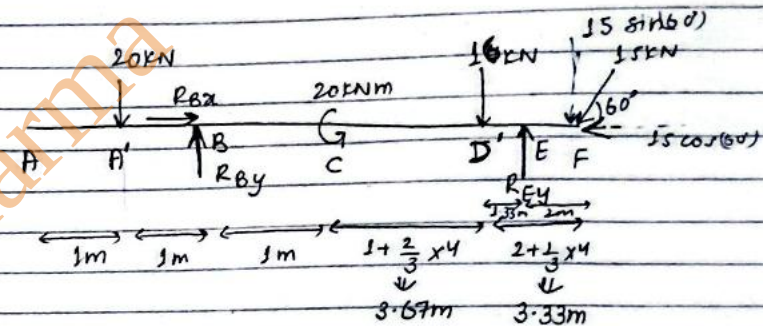
$$V_2 - V_1 = -\int w dx \quad \rightarrow w$$

Area under the load curve b/w two points always gives the change in shear force b/w those two points.

Q2



→ FBD of above Beam system is:-



Now,

Above system will be in static equilibrium when,

$$(i) \sum F_x = 0 \quad (\rightarrow +ve)$$

$$\text{or, } R_{Bx} - 15 \cos(60) = 0$$

$$\Rightarrow R_{Bx} = 7.5 \text{ N}$$

$$(ii) \sum F_y = 0 \quad (\uparrow +ve)$$

$$\text{or, } -20 + R_{By} - 16 + R_{Ey} - 15 \sin(60) = 0$$

$$\text{or, } R_{By} + R_{Ey} = 49 \quad \text{--- (1)}$$

(iii) Considering D as moment centre,
 $\sum M_D = 0$ (5 + ve)

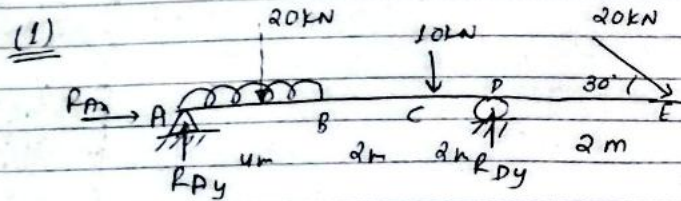
$$\text{or, } 20 \times (5.67) - R_{By} \times (4.67) + 20 + 4 \times (0) + R_{Ey} \times (1.33) - 15 \sin(60^\circ) \times 3.33 = 0$$

$$\text{or, } -4.67 R_{By} + 1.33 R_{Ey} = -90.14 \quad \text{--- (1)}$$

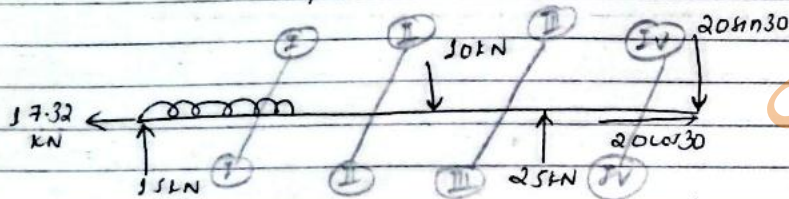
Solving (1) & (2)

$$R_{By} = 25.885 \text{ N}$$

$$R_{Ey} = 23.775 \text{ N}$$



||
 FBD of Beam



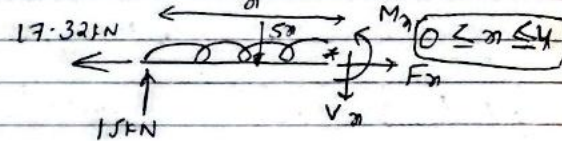
Section I-I (study points AB) ($0 \leq x \leq 4$)

For Component AB,

$$\sum F_x = 0 \quad (\rightarrow +)$$

$$-17.32 + F_x = 0$$

$$\Rightarrow F_x = 17.32 \text{ kN}$$



$$\times \sum F_y = 0 \quad (\uparrow +ve)$$

$$15 - 5x - V_x = 0$$

$$\therefore V_x = 15 - 5x$$

$$\text{For } V_x = 0, \quad x = 3 \text{ m}$$

$$\sum M_x = 0 \quad (\curvearrowright +ve)$$

$$\text{or, } 15x - (5x) \times \left(\frac{x}{2}\right) - M_x = 0$$

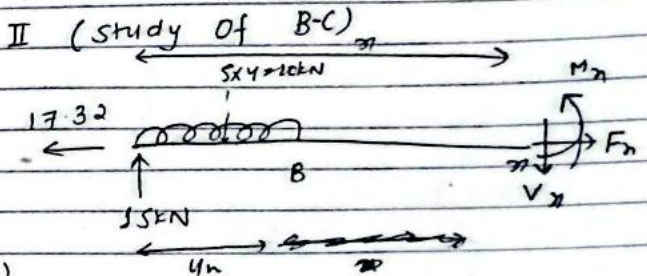
$$\Rightarrow M_x = 15x - \frac{5x^2}{2}$$

$\star \left\{ V_x \text{ is changing with } x \right\}$
 $x=0 \rightarrow 15 \text{ kN (A)}$
 just right of A
 $x=4 \rightarrow -5 \text{ kN (B)}$
 just left of B

II section II-II (study of B-C)

Study point

$$(4 \leq x \leq 6)$$



$$\times \sum F_x = 0 \quad (\rightarrow +ve)$$

$$-17.32 + F_x = 0$$

$$\Rightarrow F_x = 17.32 \text{ kN}$$

$$\times \sum F_y = 0 \quad (\uparrow +ve)$$

$$15 - (5 \times 4) - V_x = 0$$

$$\text{or, } V_x = 15 - 5 \times 4 = -5 \text{ kN}$$

$$\left\{ \begin{array}{l} x=4 \rightarrow -5 \text{ kN (B)} \\ x=6 \rightarrow -5 \text{ kN (C)} \end{array} \right.$$

$$\sum M_n = 0 \quad (\curvearrowright +ve)$$

$$15 \times x - 20 \times (x-2) - M_n = 0$$

$$\text{on } M_n = -20(x-2) + 15x$$

$$M_n = -5x + 40$$

Section III-III Study part

(6 ≤ x ≤ 8)

$$\sum F_n = 0 \quad (\rightarrow +ve)$$

$$-17.32 + F_n = 0$$

$$\Rightarrow F_n = 17.32 \text{ kN}$$

$$\sum F_y = 0 \quad (\uparrow +ve)$$

$$15 - 20 - 10 - V_n = 0$$

$$\text{on } V_n = -15 \text{ kN}$$

$$\sum M_x = 0 \quad (\curvearrowright +ve)$$

$$15x - 20(x-2) - 10(x-6) - M_n = 0$$

$$\text{on } M_n = -15x + 100$$

section - IV-IV
(8 ≤ x ≤ 10)

$$\sum F_n = 0 \quad (\rightarrow +ve)$$

$$F_n = 17.32 \text{ kN}$$

$$\sum F_y = 0 \quad (\uparrow +ve)$$

$$15 - 20 - 10 + 25 - V_n = 0$$

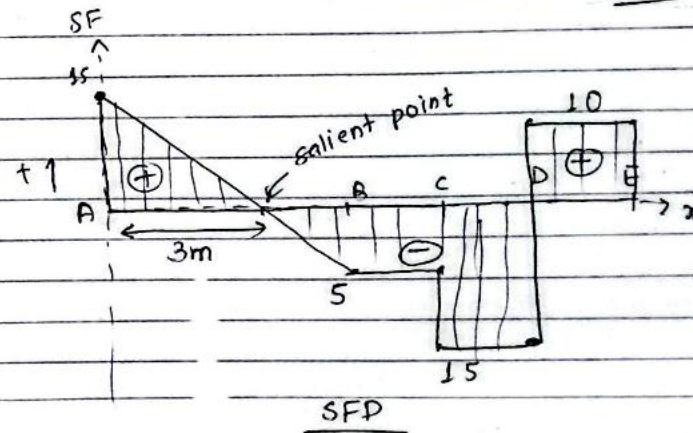
$$\text{on } V_n = +10 \text{ kN}$$

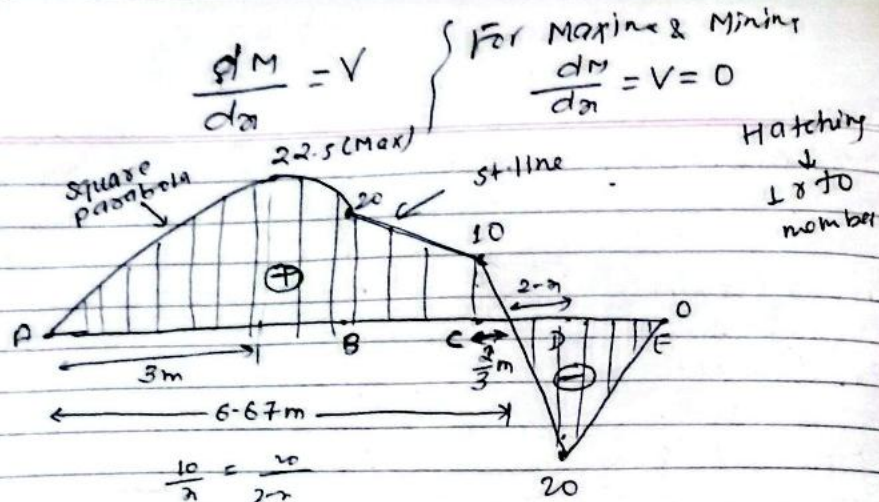
$$\sum M_x = 0 \quad (\curvearrowright +ve)$$

$$15x - 20(x-2) - 10(x-6) + 25(x-8) - M_n = 0$$

$$\text{on } 15x - 20x + 40 - 10x + 60 + 25x - 200 - M_n = 0$$

$$\text{on } M_n = 10x - 100$$





$$\frac{10}{3} = \frac{20}{2x}$$

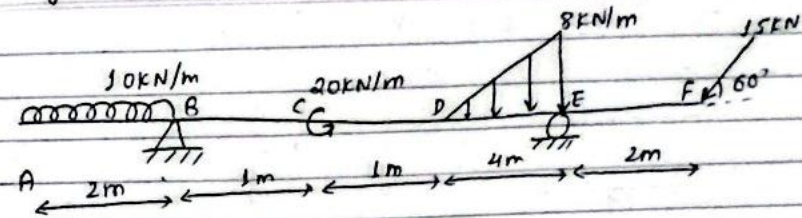
$$20 - 10x = 20x$$

$$x = \left(\frac{2}{3}\right)$$

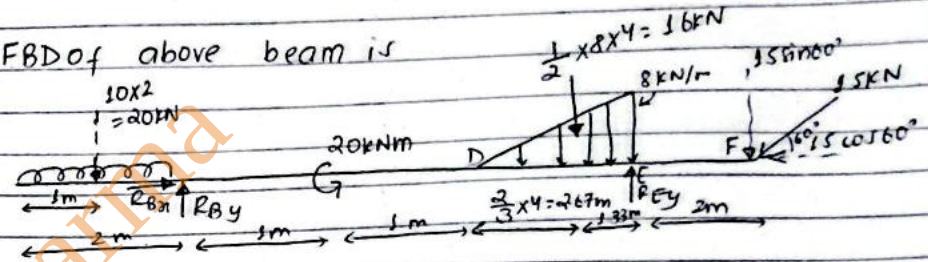
BMD

Base line $\rightarrow$ HB
Overall Diagram $\rightarrow$ 2B
Hatching $\rightarrow$ 2H

Analysis of Beam



FBD of above beam is



For stable equilibrium,

$$\sum F_x = 0 \quad (\rightarrow +ve)$$

$$\text{on } R_{Bx} - 15 \cos 60^\circ = 0$$

$$\text{or, } R_{Bx} = 7.5 \text{ kN } (\rightarrow)$$

$$\sum M_B = 0 \quad (\uparrow +ve)$$

$$\text{or, } 20 \times 1 + 20 - 16 \times 4.67 + R_{Ey} \times 6 - 15 \sin 60^\circ \times 8 = 0$$

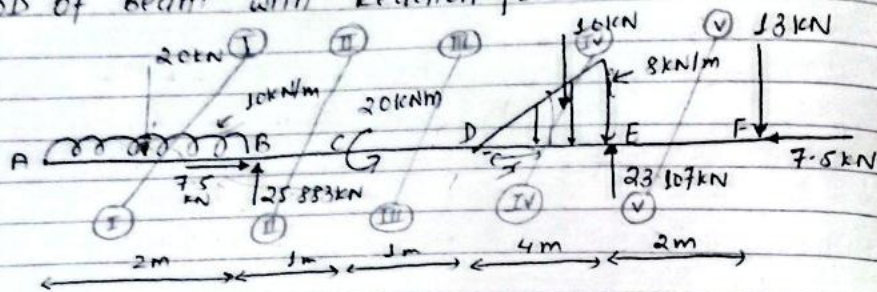
$$\text{or, } R_{Ey} = 23.107 \text{ kN } (\uparrow)$$

$$\sum F_y = 0 \quad (\uparrow +ve)$$

$$\text{on, } -20 + R_{By} - 16 + R_{Ey} - 15 \sin 60^\circ = 0$$

$$\text{on, } R_{By} = 25.883 \text{ kN } (\uparrow)$$

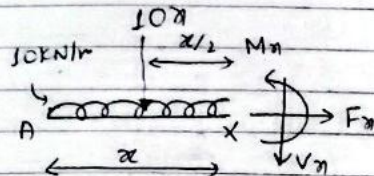
FBD of Beam with Reaction forces value:



Dividing Beams into sections

For section I-I
(study of A-B)

$$0 \leq x \leq 2$$



$$\sum F_x = 0 \quad (\rightarrow +ve)$$

or, $F_x = 0kN$

$x=0 \rightarrow 0kN$ (Just right of 'A')

$x=2 \rightarrow 0kN$ (Just left of 'B')

$$\sum F_y = 0 \quad (\uparrow +ve)$$

or, $-10x + V_x = 0$

or, $V_x = -10x$

$x=0 \rightarrow 0kN$ (Just right of 'A')

$x=2 \rightarrow -20kN$ (Just left of 'B')

$$\sum M_x = 0 \quad (\curvearrowright +ve)$$

or, $-10x \times \frac{x}{2} - M_x = 0$

or, $M_x = -5x^2$

$x=0 \rightarrow 0kNm$ (Just right of 'A')

$x=2 \rightarrow -20kNm$ (Just left of 'B')

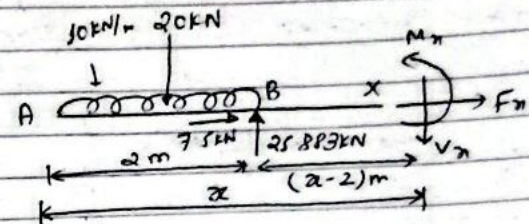
For section II-II

Study of B-C
($2 \leq x \leq 3$)

$$\sum F_x = 0 \quad (\rightarrow +ve)$$

or, $7.5 + F_x = 0$

or, $F_x = -7.5kN$



$x=2 \rightarrow -7.5kN$ (B)

$x=3 \rightarrow -7.5kN$ (C)

$$\sum F_y = 0 \quad (\uparrow +ve)$$

or, $-20 + 25.883 - V_x = 0$

or, $V_x = 5.883kN$

$x=2 \rightarrow 5.883kN$ (B)

$x=3 \rightarrow 5.883kN$ (C)

$$\sum M_x = 0 \quad (\curvearrowright +ve)$$

or, $-20 \cdot (x-1) + 25.883(x-2) - M_x = 0$

or, $M_x = 5.883x - 31.766$

{Eqⁿ of S.L. line}

$x=2 \rightarrow -20kNm$ (Just right of 'B')

$x=3 \rightarrow -14.117kNm$ (Just left of 'C')

For section III-III

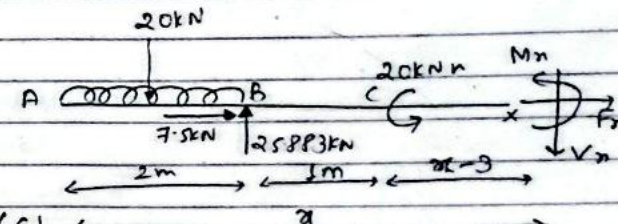
Study of C-D

$$(3 \leq x \leq 4)$$

$$\sum F_x = 0 \quad (\rightarrow +ve)$$

or, $7.5 + F_x = 0$

or, $F_x = -7.5kN$



$x=3 \rightarrow -7.5kN$ (C)

$x=4 \rightarrow -7.5kN$ (D)

$$\sum F_y = 0 \quad (\uparrow +ve)$$

$$\text{or } -20 + 25.883 - V_m = 0$$

$$\text{or } V_m = 5.883 \text{ kN}$$

$$x=3 \rightarrow 5.883 \text{ kN (C)}$$

$$x=4 \rightarrow 5.883 \text{ kN (D)}$$

$$\sum M_x = 0 \quad (\curvearrowright +ve)$$

$$\text{or } -20 \times (x-1) + 25.883 \times (x-2) - 20 - M_m = 0$$

$$\text{or } -20x + 20 + 25.883x - 51.766 - 20 - M_m = 0$$

$$\text{or } M_m = 5.883x - 51.766$$

{eqn of
str line}

$$x=3 \rightarrow -34.117 \text{ kNm (Just right of 'C')}$$

$$x=4 \rightarrow -28.234 \text{ kNm}$$

(Just left of 'D')

For Section IV-IV

Study of D-E

$$4 \leq x \leq 8$$

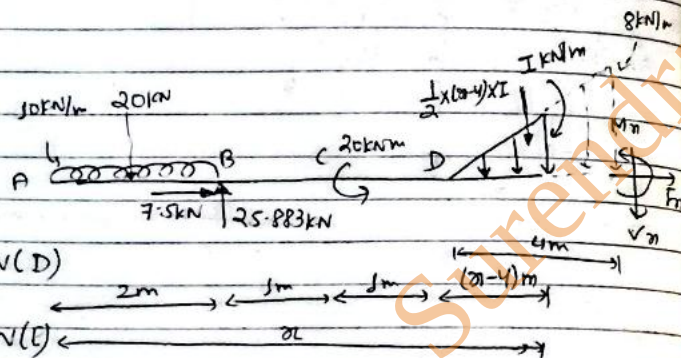
$$\sum F_x = 0$$

$$\text{or } 7.5 + F_m = 0$$

$$\text{or } F_m = -7.5 \text{ kN}$$

$$x=4 \rightarrow -7.5 \text{ kN (D)}$$

$$x=8 \rightarrow -7.5 \text{ kN (E)}$$



For UVL

From congruency of triangle

$$\frac{x-4}{4} = \frac{I}{8}$$

$$\text{or } I = 2x - 8 \text{ kN/m}$$

↳ Intensity of UVL from
a distance from A

$$\sum F_y = 0 \quad (\uparrow +ve)$$

$$\text{or } -20 + 25.883 - \frac{1}{2}(x-4) \times I - V_m = 0$$

$$\text{or } 5.883 - \frac{1}{2}(x-4)(2x-8) - V_m = 0$$

$$\text{or } V_m = 5.883 - (x-4)^2$$

$$\text{or } V_m = 5.883 - (x^2 - 8x + 16)$$

$$\text{or } V_m = 5.883 - x^2 + 8x - 16$$

$$\text{or } V_m = -x^2 + 8x - 10.117$$

{eqn of square
parabola}

$$x=4 \rightarrow 5.883 \text{ kN (D)}$$

For $V_m = 0$, $x = 6.425 \text{ m}$

$$x=8 \rightarrow -10.117 \text{ kN (E)}$$

(Just left of 'E')

$$x=5 \rightarrow 4.883 \text{ kN}$$

$$\sum M_x = 0 \quad (\curvearrowright +ve)$$

$$\text{or } -20 \times (x-1) + 25.883 \times (x-2) - 20 - \frac{1}{2}(x-4)(2x-8) \times \frac{1}{3}(x-4) - M_m = 0$$

$$- M_m = 0$$

$$\text{or } -20x + 20 + 25.883x - 51.766 - 20 - \frac{1}{3}(x-4)^3 - M_m = 0$$

$$\text{or } M_m = -\frac{1}{3}(x-4)^3 + 5.883x - 51.766$$

{eqn of
cubic parabola}

$$x=4 \rightarrow -28.234 \text{ kNm}$$

(Just right of 'D')

$$x=8 \rightarrow 18.7212 \text{ kNm}$$

(Just left of 'E')

$$x=6.425 \rightarrow -18.7212 \text{ kNm}$$

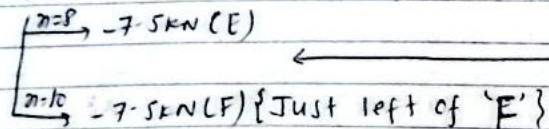
For section I-I

For study of E-F
 $8 \leq x \leq 10$

$$\sum F_x = 0 \quad (\rightarrow +ve)$$

$$\text{or } 7.5 + F_x = 0$$

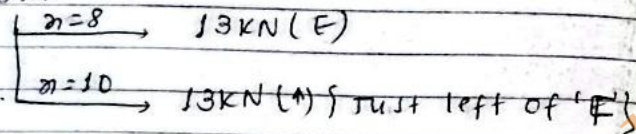
$$\text{or } F_x = -7.5 \text{ kN}$$



$$\sum F_y = 0 \quad (\uparrow +ve)$$

$$\text{or } -20 + 25.883 - 16 + 23.107 - V_x = 0$$

$$\text{or } V_x = 13 \text{ kN}$$



$$\sum M_x = 0 \quad (\uparrow +ve)$$

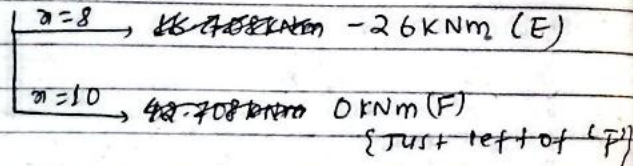
$$\text{or } -20 \times (x-1) + 25.883(x-2) - 20 - 16 \left(x - \frac{4}{3} - \frac{2}{3} \times 4 \right)$$

$$+ 23.107(x-8) - M_x = 0$$

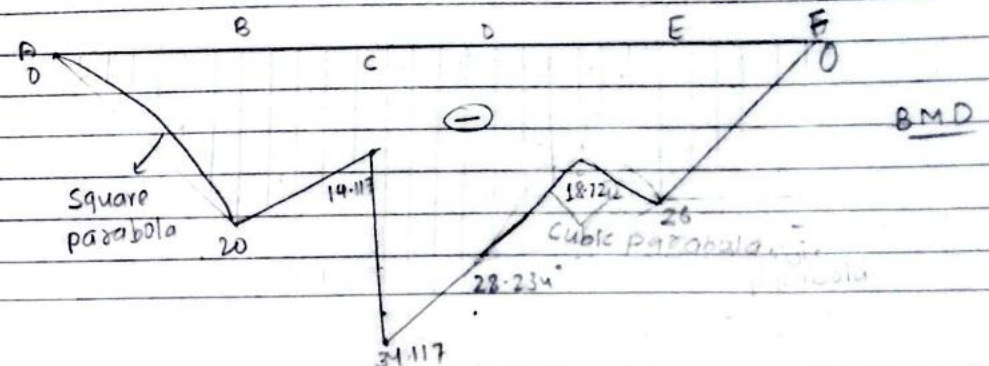
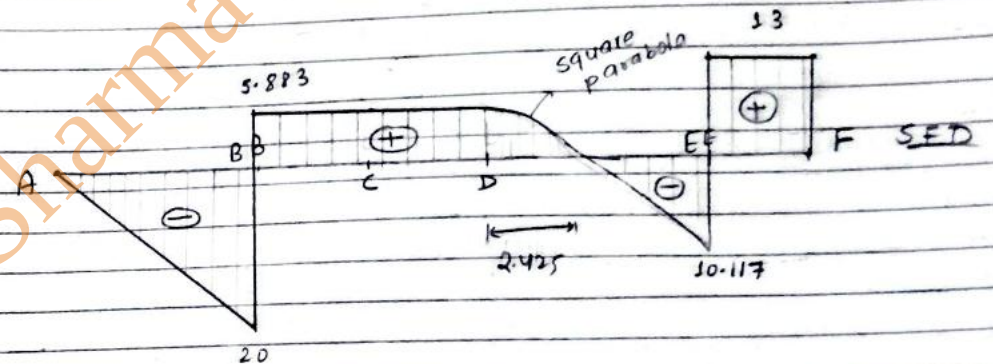
$$\text{or } -20x + 20 + 25.883x - 51.766 - 20 - 16x + 106.67$$

$$+ 23.107x - 184.856 - M_x = 0$$

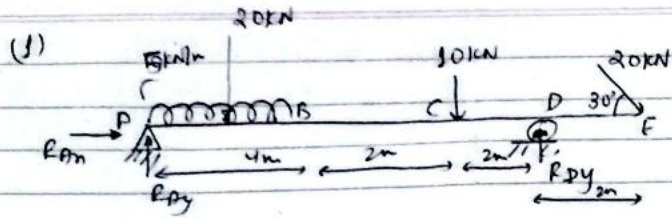
$$\text{or } M_x = 13x - 87.249$$



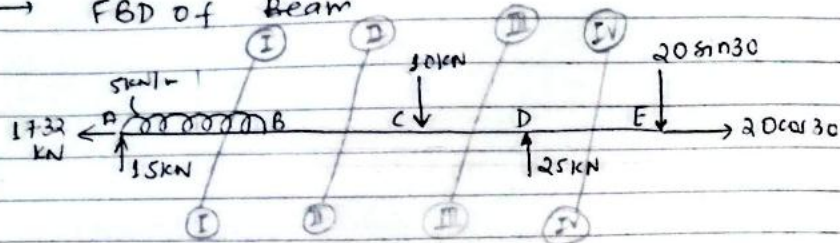
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$$\frac{dM}{dn} = \frac{V}{0}$$



→ FBD of Beam



Analysis from right to left

Section IV-IV

Study of E-D

$$0 \leq x \leq 2$$

$$\sum F_x = 0 (\rightarrow +ve)$$

$$\begin{aligned} \text{or } -F_n + 17.32 &= 0 \\ \text{or } F_n &= 17.32 \text{ kN} \end{aligned}$$

$x=0 \rightarrow 17.32 \text{ kN}$
 $x=2 \rightarrow 17.32 \text{ kN}$

$$\sum F_y = 0 (\uparrow +ve)$$

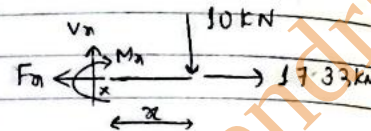
$$\begin{aligned} \text{or } V_n - 10 &= 0 \\ \text{or } V_n &= 10 \text{ kN} \end{aligned}$$

$x=0 \rightarrow 10 \text{ kN}$
 $x=2 \rightarrow 10 \text{ kN}$

$$\sum M_x = 0 (\curvearrowright +ve)$$

$$\begin{aligned} \text{or } M_n + 10x &= 0 \\ \text{or } M_n &= -10x \end{aligned}$$

$x=0 \rightarrow 0 \text{ kN}$
 $x=2 \rightarrow -20 \text{ kN}$



Section III-III

Study of D-C

$$2 \leq x \leq 4$$

$$\sum F_x = 0 (\rightarrow +ve)$$

$$\text{or } -F_n + 17.32 = 0$$

$$\text{or } F_n = 17.32 \text{ kN}$$

$$\begin{aligned} x=2 &\rightarrow 17.32 \text{ kN} \\ x=4 &\rightarrow 17.32 \text{ kN} \end{aligned}$$

$$\sum F_y = 0 (\uparrow +ve)$$

$$\text{or } V_n + 25 - 10 = 0$$

$$\text{or } V_n = -15 \text{ kN}$$

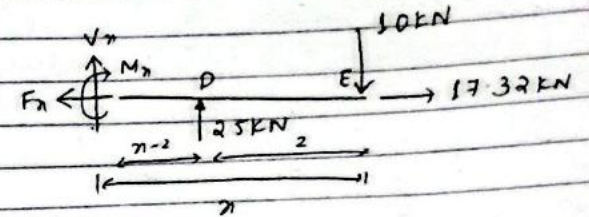
$$\begin{aligned} x=2 &\rightarrow -15 \text{ kN} \\ x=4 &\rightarrow -15 \text{ kN} \end{aligned}$$

$$\sum M_x = 0 (\curvearrowright +ve)$$

$$\text{or } M_n - 25(x-2) + 10x = 0$$

$$\text{or } M_n = 15x - 50$$

$$\begin{aligned} x=2 &\rightarrow -20 \text{ kN} \\ x=4 &\rightarrow 10 \text{ kN} \end{aligned}$$



Section II-II

Study of C-B

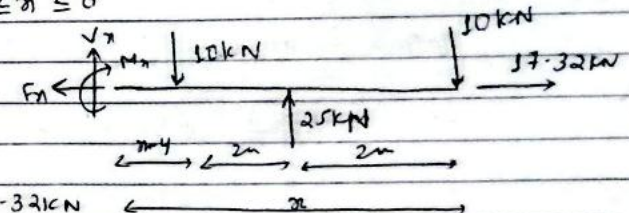
$$4 \leq x \leq 6$$

$$\sum F_x = 0 (\rightarrow +ve)$$

$$\text{or } -F_n + 17.32 = 0$$

$$\text{or } F_n = 17.32 \text{ kN}$$

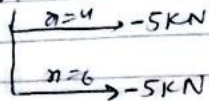
$$\begin{aligned} x=4 &\rightarrow 17.32 \text{ kN} \\ x=6 &\rightarrow 17.32 \text{ kN} \end{aligned}$$



$$\sum F_y = 0 \quad (\uparrow +ve)$$

$$\text{or } V_x - 10 - 10 + 25 = 0$$

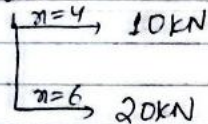
$$\text{or } V_x = -5 \text{ kN}$$



$$\sum M_x = 0 \quad (\curvearrowright +ve)$$

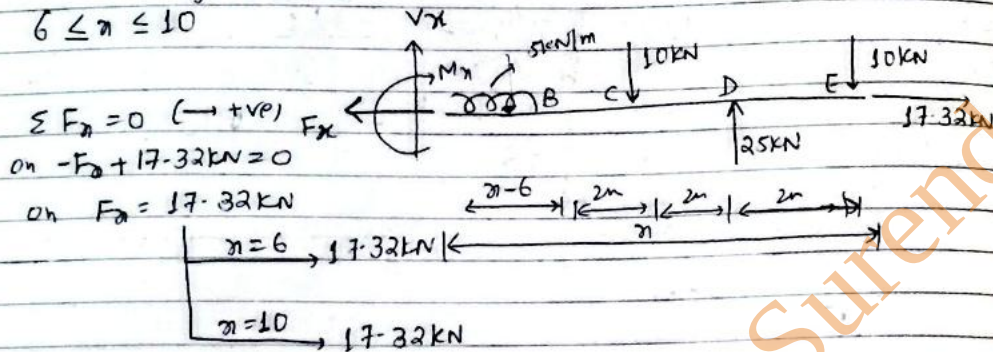
$$\text{or } M_x + 10(x-4) - 25(x-2) + 10x = 0$$

$$\text{or } M_x = 5x - 10$$



Section J-I
study of B-A

$$6 \leq x \leq 10$$



$$\sum F_x = 0 \quad (\rightarrow +ve)$$

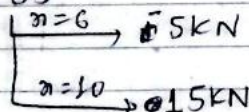
$$\text{or } -F_x + 17.32 \text{ kN} = 0$$

$$\text{or } F_x = 17.32 \text{ kN}$$

$$\sum F_y = 0 \quad (\uparrow +ve)$$

$$\text{or } V_x - 5(x-6) - 10 + 25 - 10 = 0$$

$$\text{or } V_x = 5x - 35$$

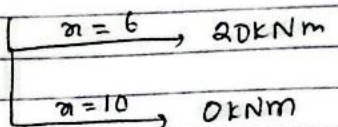


$$\sum M_x = 0 \quad (\curvearrowright +ve)$$

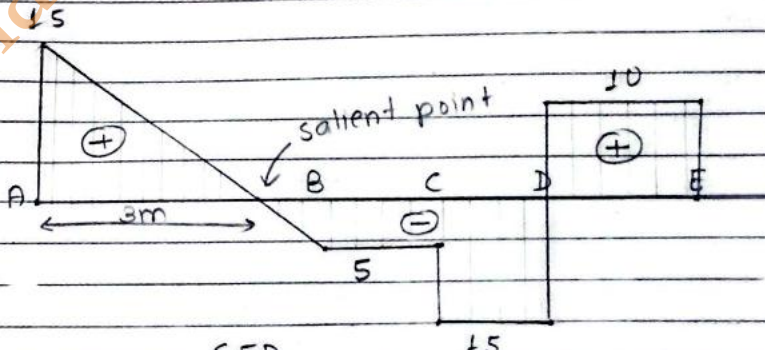
$$\text{or } M_x + 5(x-6)(x-6) + 10(x-4) - 25(x-2) + 10(x-2) = 0$$

$$\text{or } M_x = 25(x-2) - 10x - 10(x-4) - \frac{5}{2}(x-6)^2$$

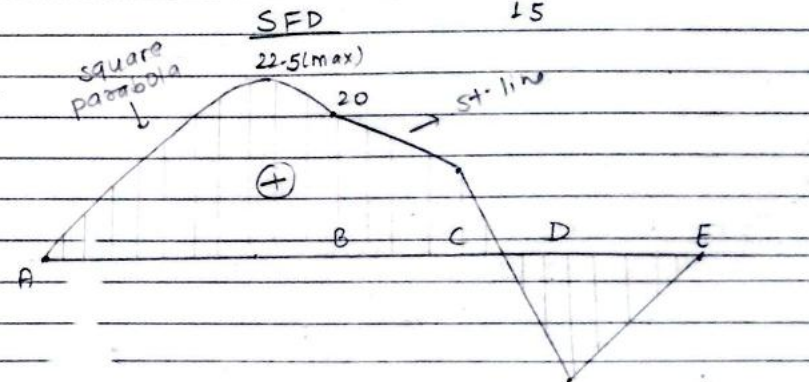
$$M_x = 5x - \frac{5}{2}(x-6)^2 - 10$$



#



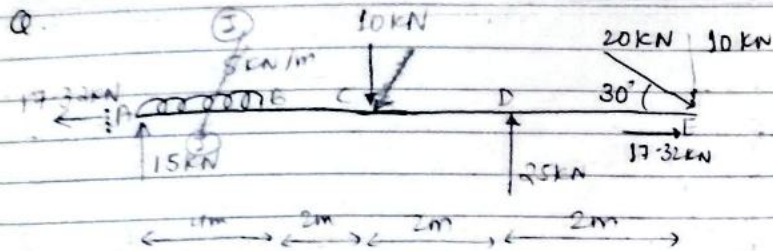
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Nature of curves

	SFD	BMD
Point Load	St. line (horizontal)	St. line (Inclined)
UDL	St. line (Inclined)	Square Parabola
UVL	Square Parabola	Cubic Parabola

Shortcut



Axial Force

$$F_{AB} = 17.32 \text{ kN (Tensile)}$$

$$F_{BC} = 17.32 \text{ kN (Tensile)}$$

$$F_{CD} = 17.32 \text{ kN (Tensile)}$$

$$F_{DE} = 17.32 \text{ kN (Tensile)}$$

$$F_{AE} = 17.32 \text{ kN (Tensile)}$$



During calculation of SF, if we encounter point load the SF values right & left of that point will be different else, it will be same.

Shear Force calculation (L to R) $\uparrow$ +ve

$$(V_A)_L = 0$$

$$(V_A)_R = 15 \text{ kN}$$

$$(V_B)_L = 15 - 5 \times 4 = -5 \text{ kN}$$

$$(V_C)_L = 15 - 5 \times 4 = -5 \text{ kN}$$

$$(V_C)_R = (V_C)_L - 10 = -5 - 10 = -15 \text{ kN}$$

$$(V_D)_L = (V_C)_R = -15 \text{ kN}$$

$$(V_D)_R = (V_D)_L + 25 = -15 + 25 = 10 \text{ kN}$$

$$(V_E)_L = (V_D)_R = 10 \text{ kN}$$

$$(V_E)_R = (V_E)_L - 10 = 0 \text{ kN}$$

For section I-I

Study of A-B

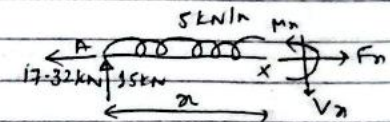
$$0 \leq x \leq 4$$

$$V_x = 15 - 5x$$

For,

$$V_x = 0$$

$$x = 3 \text{ m from A}$$



Bending Moment calculation (L to R) $\curvearrowright$ +ve

$$M_A = 0$$

$$M_B = 15 \times 4 - (5 \times 4) \times 2 = 20 \text{ kNm}$$

$$M_C = 15 \times 8 - (5 \times 4) \times 4 = 10 \text{ kNm}$$

$$M_D = 15 \times 8 - 5 \times 4 \times 6 - 10 \times 2 = -20 \text{ kNm}$$

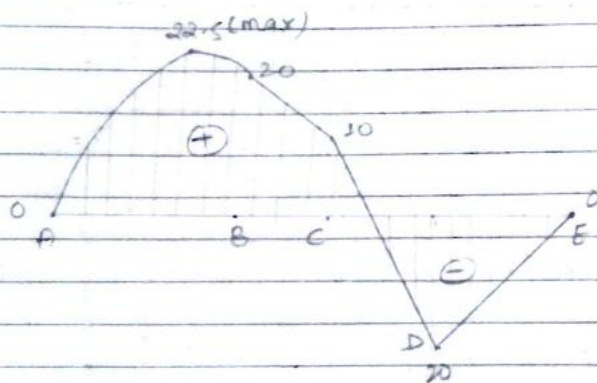
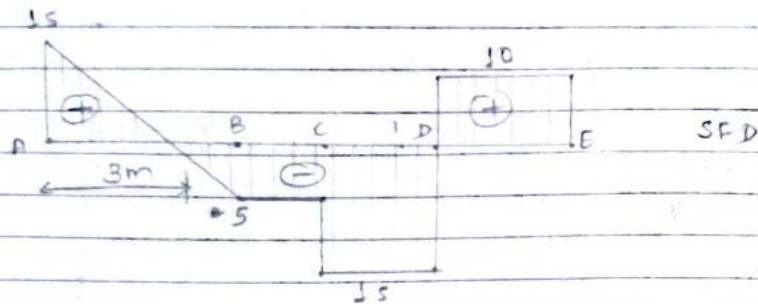
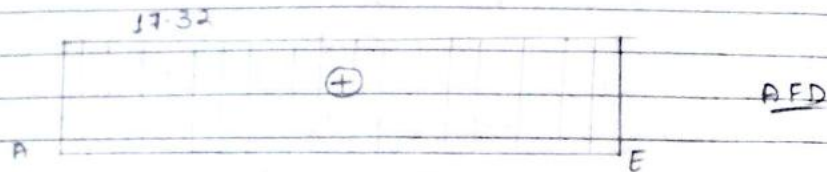
$$M_E = 15 \times 10 - 5 \times 4 \times 8 - 10 \times 4 + 25 \times 2 = 0 \text{ kNm}$$

For section I-I

$$M_x = 15x - 5x \times \frac{x}{2} = 15x - 2.5x^2$$

$$\text{At } x = 3, M_x = 22.5 \text{ kNm}$$

- ★ During Bending moment, calculation unless we encounter couple Bending moment left & right of that point remains same.
Else, couple should be taken into consideration (if couple present at the point)



- we there were four unknowns
But, taking eqns which solved two variables
eg: $(\sum M)_{\text{right of F}} = 0$ & $M_A = 0$
L we found value of R_{Ax} & R_{Ay}
with help of sem. eqns we found remaining unknowns

(a) Analyse the Frame

→ Calculation of Unknown Forces

(i) $(\sum M)_{\text{right of F}} = 0$ (↺ +ve)

$$\begin{aligned} \text{or, } (10 \times 3) \times 1.5 + 30 \\ + 10 \times 2 - R_{Ay} \times 4 \\ + R_{Ax} \times 3 + 10 \sin(60^\circ) \times 4.5 = 0 \end{aligned}$$

$$\text{or, } 134 - 4R_{Ay} - 3R_{Ax} = 0$$

$$\text{or, } 4R_{Ax} + 3R_{Ay} = 134 \quad \text{--- (i)}$$

(ii) $\sum F_x = 0$ (↔ +ve)

$$R_{Ax} + \frac{1}{2} \times 3 \times 15 + 20 \cos 30^\circ - 10 \cos 60^\circ - 10 + R_{Ay} = 0$$

$$\text{or, } R_{Ax} + R_{Ay} = -24.82 \quad \text{--- (ii)}$$

(iii) $\sum F_y = 0$ (↓ +ve)

$$-R_{Ay} + 20 \sin 30^\circ + 10 \times 3 + 10 \sin 60^\circ - R_{Ay} = 0$$

$$\text{or, } R_{Ay} + R_{Ay} = 48.66 \quad \text{--- (iii)}$$

(iv) $\sum M_A = 0$ (↺ +ve)

$$\begin{aligned} \text{or, } 22.5 \times (1 + \frac{2}{3} \times 3) + 20 \cos 30^\circ \times 6 - 25 + (10 \times 3) \times (4.5) \\ + 30 + 10 \sin 60^\circ \times 7.5 \\ - 10 \cos 60^\circ \times 6 - 10 \times 4 \end{aligned}$$

$$\text{or, } 2R_{Ax} - 6R_{Ay} = -306.37 \quad \text{--- (iv)}$$

Solving eqⁿ ① & ②

$$R_{Tn} = 63.95 \text{ kN} - 3.837 \text{ kN} = 3.837 \text{ kN} (\leftarrow)$$

$$R_{Ty} = 53.19 \text{ kN} \quad 49.78 \text{ kN} (\uparrow)$$

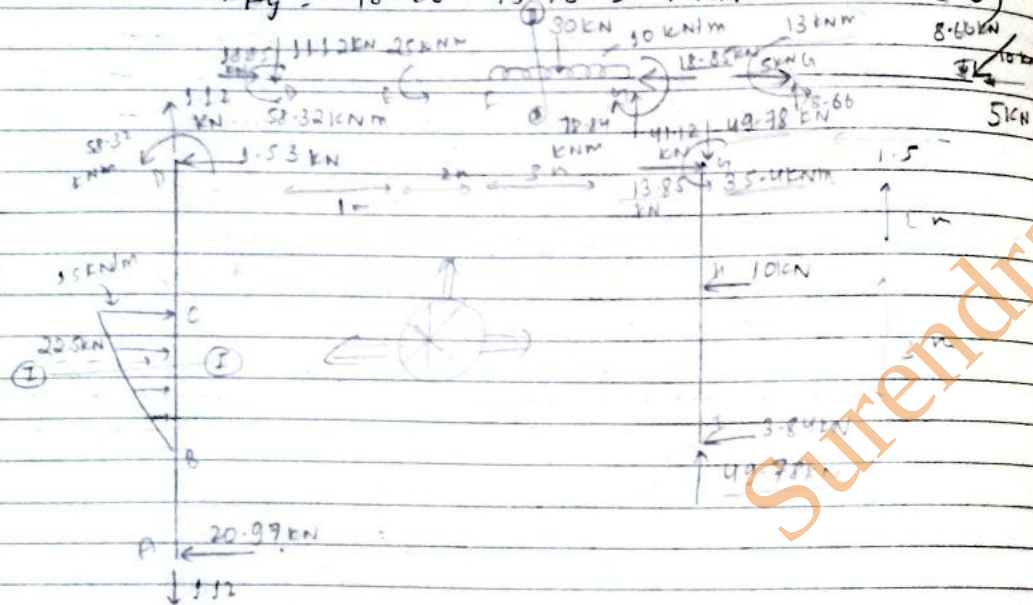
From ③

$$R_{Pn} = -24.82 + 3.837 = -20.983 \text{ kN} = 20.983 \text{ kN} (\leftarrow)$$

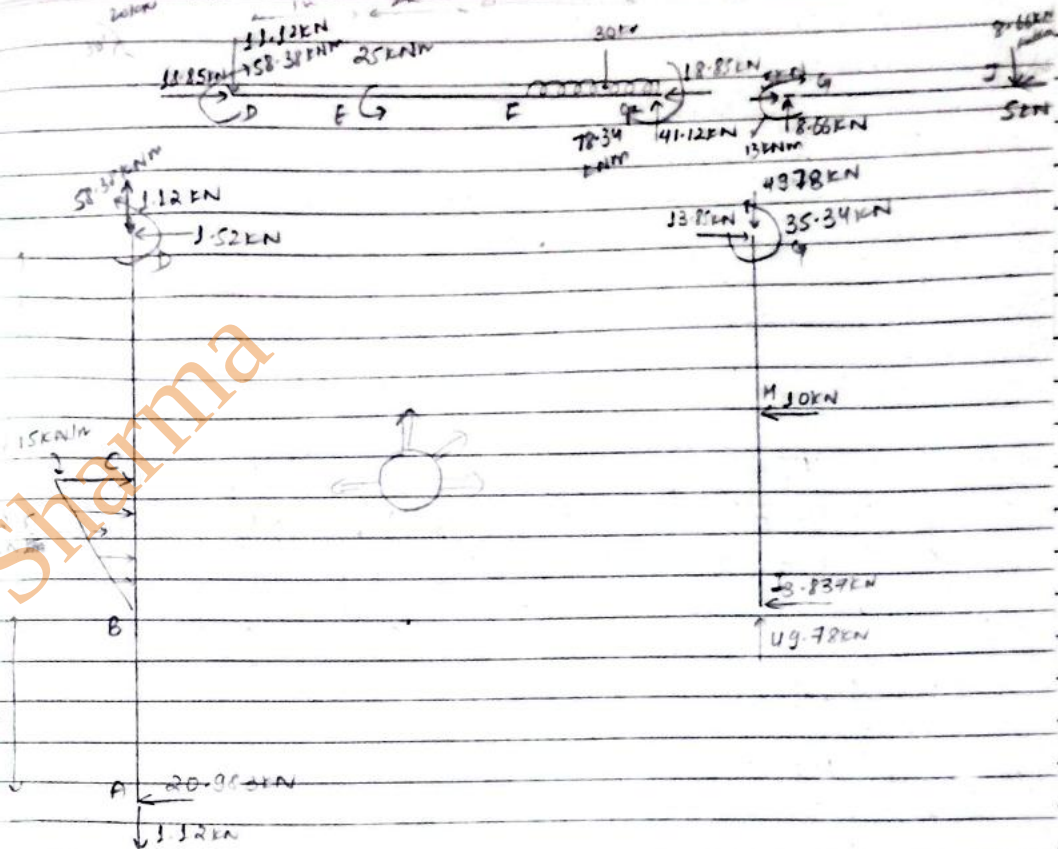
$$= -20.983$$

$$= 20.983 (\leftarrow)$$

$$R_{Py} = 48.66 - 49.78 = -1.12 \text{ kN} = 1.12 \text{ kN} (\downarrow)$$



Redrawn figure:-



Axial forces

For member AB,

$$F_{AB} = 1.12 \text{ kN (Tensile)}$$

For member DG,

$$F_{DG} = -18.85 \text{ kN (Compressive)}$$

For member GI,

$$F_{GI} = -49.78 \text{ kN (Compressive)}$$

For member GJ,

$$F_{GJ} = -5 \text{ kN (Compressive)}$$

Shear force calculation

For member AD (L → R ↑ +ve)

$$(V_D)_L = 0 \text{ kN}$$

$$(V_D)_R = 20.97 \text{ kN}$$

$$(V_B) = 20.97 \text{ kN}$$

$$V_C = 20.97 - 22.5 = -1.53 \text{ kN}$$

$$(V_D)_L = V_C = -1.53 \text{ kN}$$

$$(V_D)_R = V_C + 1.53 \text{ kN} = 0 \text{ kN}$$

For section I-I,

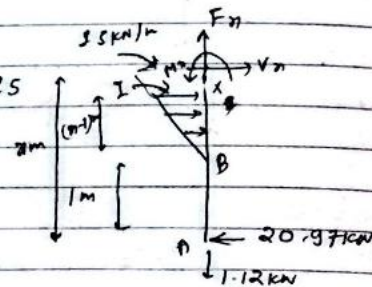
$$1 \leq x \leq 4$$

$$I = \frac{(x-1)}{3} \times 15$$

$$V_x = 20.97 - \frac{1}{2} \times 15 \times (x-1)$$

$$V_x = -2.5(x-1)^2 + 20.97$$

$$\text{For } V_x = 0, x = 3.896 \text{ m}$$



For member DG,

$$(V_D)_L = 0$$

$$(V_D)_R = -11.12 \text{ kN}$$

$$V_E = -11.12 \text{ kN}$$

$$V_F = -11.12 \text{ kN}$$

$$(V_G)_L = -11.12 - 30 = -41.12 \text{ kN}$$

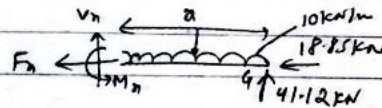
$$(V_G)_R = -41.12 + 41.12 = 0 \text{ kN}$$

For section II-II

$$0 \leq x \leq 3 \quad (R \rightarrow L \downarrow +ve)$$

$$V_x = -41.12 + 10x$$

$$\text{For } V_x = 0 \Rightarrow x = 4.112 \text{ m}$$



$$= \left[\frac{7^3 \times 5}{36} + 17.5(3.71 - 4.67)^2 \right]$$

$$+ \left[\frac{\pi \times 3^4}{8} + 14.14(3.71 - 3)^2 \right]$$

$$- \left[\frac{1^3 \times 5}{36} + 2.5(3.71 - 6.67)^2 \right]$$

$$- \left[\frac{\pi (0.5)^4}{4} + 0.78(3.71 - 3)^2 \right]$$

$$= 63.77 + 38.94 - 22.04 - 0.44$$

$$= 80.23 \text{ m}^4$$

Q. "Objects do not necessarily need to be small to be accurately idealized as a particle in the study of mechanics." Justify the statement with suitable example.

→ Particle is defined as matter whose dimension can be neglected in comparison to other dimensions involved in the system. It is nothing but "point mass" (i.e. whole mass is assumed to be ~~center~~ concentrated through a single point or force applied on it are assumed to be concentrated to a single point).

In Micro analysis, particles are smaller than an atom but in macro analysis, particles are larger than earth.

Example:- Size Though reference to the masses on earth the size of earth is significantly large.

But, size of earth is insignificant compared to the size of its orbit in macro analysis.

Thus, earth can be modeled as a particle when studying its orbital motion.

Thus, object do not necessarily...

→ Different laws can be applied on rigid body for example the principle of transmissibility.

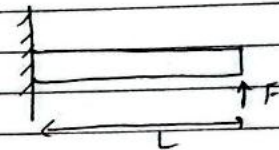
Rigid body

→ It is the combination of two or more particles which are so interconnected that they do not change their relative position however large force applied maybe.

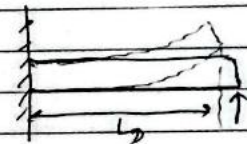
→ In general, all bodies change their shape and size on application of force. But, for solid bodies, it is very small, so solid body are assumed to be perfectly rigid. in study of applied Mechanics [We don't have "perfectly rigid body"]

→ Deformations in solid body are comparatively lesser than fluid or deformable bodies.

If, the body is deformable, then forces applied change the geometry and thus the actual moment present (as moment arm length varies if body deforms)



Rigid beam



Deformable beam

$$L_0 < L$$

For member GJ

$$(V_G)_L = 0 \text{ kN}$$

$$(V_G)_R = 8.66 \text{ kN}$$

$$(V_J)_L = 8.66 \text{ kN}$$

$$(V_J)_R = 8.66 - 8.66 = 0 \text{ kN}$$

For member GI,

$$(V_G)_L = 0 \text{ kN}$$

$$(V_G)_R = 3.85 \text{ kN}$$

$$(V_I)_L = 3.85 \text{ kN}$$

$$(V_I)_R = 3.85 - 10 = -6.15 \text{ kN}$$

$$(V_J)_L = 8.85 \text{ kN}$$

$$(V_J)_R = 8.85 - 8.84 = 0.01 \text{ kN}$$

Bending moment calculation (L → R → (+ve))
(R → L → (-ve))

For member AD,

$$M_A = 0 \text{ kNm}$$

$$M_B = 20.97 \times 1 = 20.97 \text{ kNm}$$

$$M_C = 20.97 \times 4 - 22.5 \times 1 = 61.38 \text{ kNm}$$

$$(M_D)_L = 20.97 \times 6 - 22.5 \times 3 = 58.32 \text{ kNm}$$

$$(M_D)_R = -(M_D)_L - 58.32 = 0 \text{ kNm}$$

For section I-I

$$1 \leq x \leq 4$$

$$M_x = 20.97 \times x - \frac{1}{2} \times 5(x-1)(x-1) \times \frac{(x-1)}{3}$$

$$M_x = -\frac{5}{6}(x-1)^3 + 20.97x$$

$$x = 3.846 \text{ m}, \quad M_x = 61.46 \text{ kNm}$$

For member DG,

$$(M_D)_L = 0$$

$$(M_D)_R = 58.32 \text{ kNm}$$

$$(M_E)_L = 58.32 - 11.12 \times 1 = 47.2 \text{ kNm}$$

$$(M_E)_R = 47.2 - 25 = 22.2 \text{ kNm}$$

$$M_F = 58.32 - 11.12 \times 3 - 25 = -0.04 \text{ kNm}$$

$$(M_G)_L = 58.32 - 11.12 \times 6 - 25 - 30 \times 1.5 = -78.88 \text{ kNm}$$

$$(M_G)_R = (M_G)_L + 78.84 = -0.04 \text{ kNm}$$

For section II-II,

$$0 \leq x < 3$$

$$M_x = 41.12x + 10 \frac{x^2}{2} = 20.5x^2 + 41.12x$$

For member GD

$$(M_G)_L = 0$$

$$(M_G)_R = -13 \text{ kNm}$$

$$(M_J)_L = -13 + 8.66 \times 1.5 = 0 \text{ kNm}$$

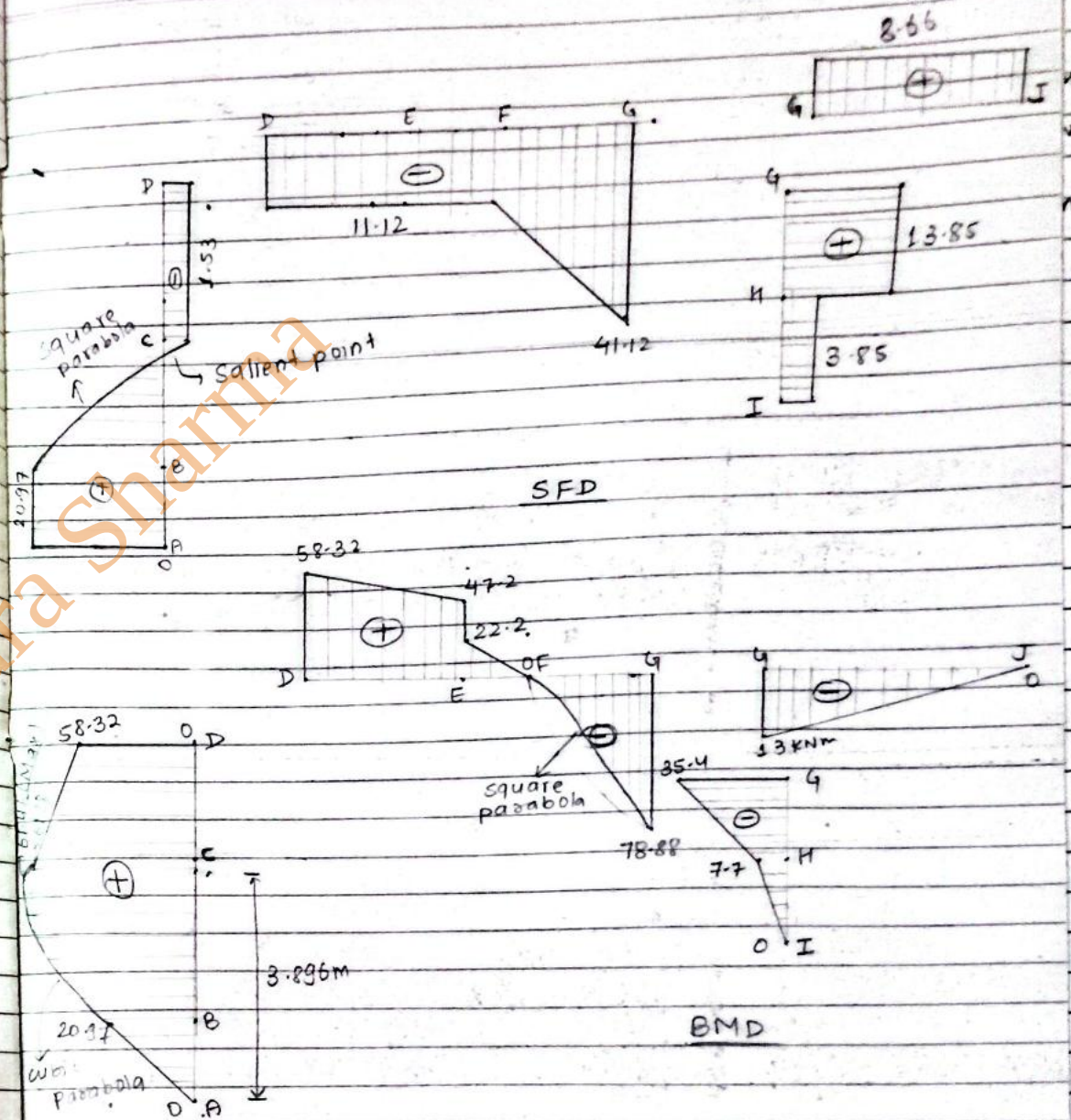
For member GI,

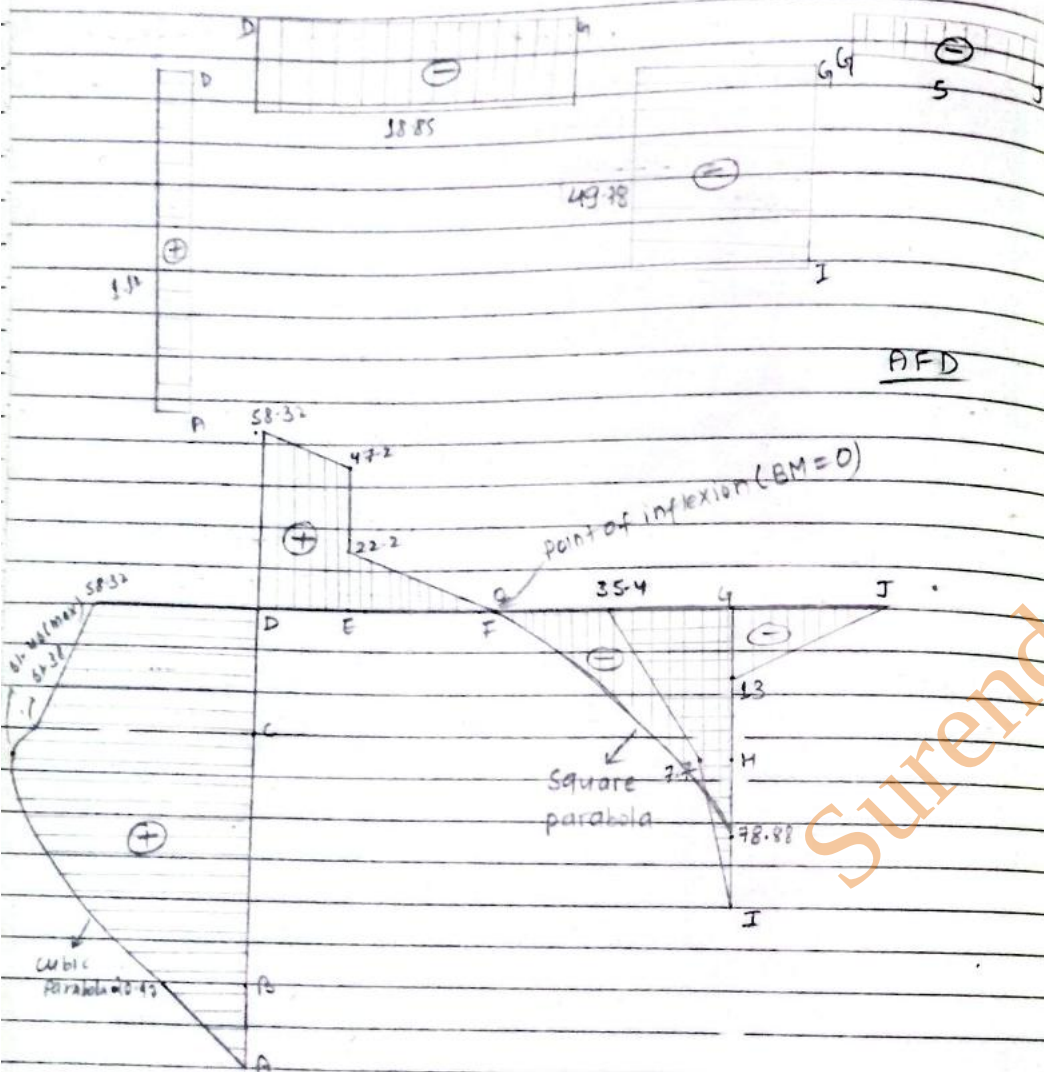
$$(M_G)_L = 0 \text{ kNm}$$

$$(M_G)_R = -35.4 \text{ kNm}$$

$$M_H = -35.4 + 13.85 \times 2 = -7.7 \text{ kNm}$$

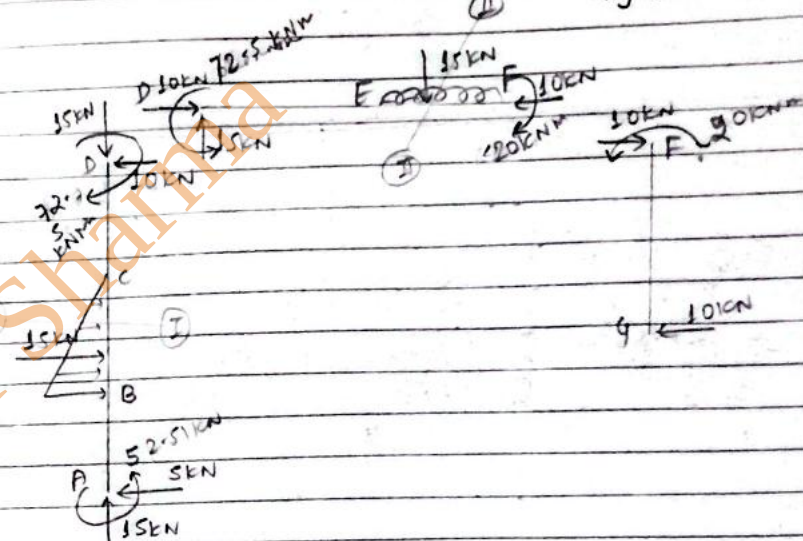
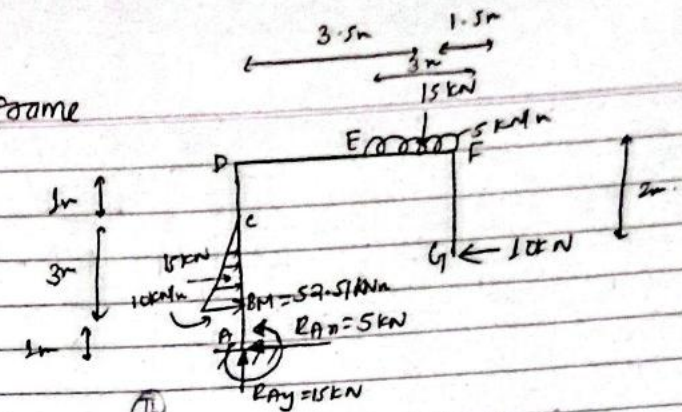
$$M_I = -35.4 + 13.85 \times 4 - 10 \times 2 = 0 \text{ kNm}$$





AFD

Q.3 Analyze the frame



For member AB,

Fixed shear force,

$$(V_A)_L = 0$$

$$(V_A)_R = 5 \text{ kN}$$

$$V_B = 5 \text{ kN}$$

$$V_C = 5 - 15 = -10 \text{ kN}$$

$$V_D = -10 + 10 = 0 \text{ kN}$$

Bending Moment,

$$(M_A)_L = 0$$

$$(M_A)_R = -52.51 \text{ kNm}$$

$$M_B = -52.51 + 5 \times 1 = -47.51 \text{ kNm}$$

$$M_C = -52.51 + 5 \times 4 = -15 \times 2 = -62.51 \text{ kNm}$$

$$(M_D)_L = -52.51 + 5 \times 5 - 15 \times 3 = -72.51 \text{ kNm}$$

$$(M_D)_R = -72.51 + 72.51 = 0 \text{ kNm}$$

For section I-I

$$1 \leq x \leq 4$$

$$I = \left(\frac{5}{x^2} \right) \times 10$$

$$V_x = -10 + \frac{10}{2} (x-1)^{\frac{2-1}{3}}$$

$$V_x = \frac{5}{3} (x-1)^2 - 10$$

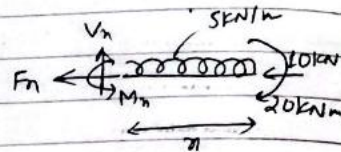
$$\text{At } V_x = 0, x = 3.45 \text{ m}$$

$$M_x = -72.51 + 10x^2 + \frac{10}{6} (x-1)^{\frac{2-1}{3}}$$

$$M_x = \frac{-10}{3} (x-1)^3 + 10x - 72.51$$

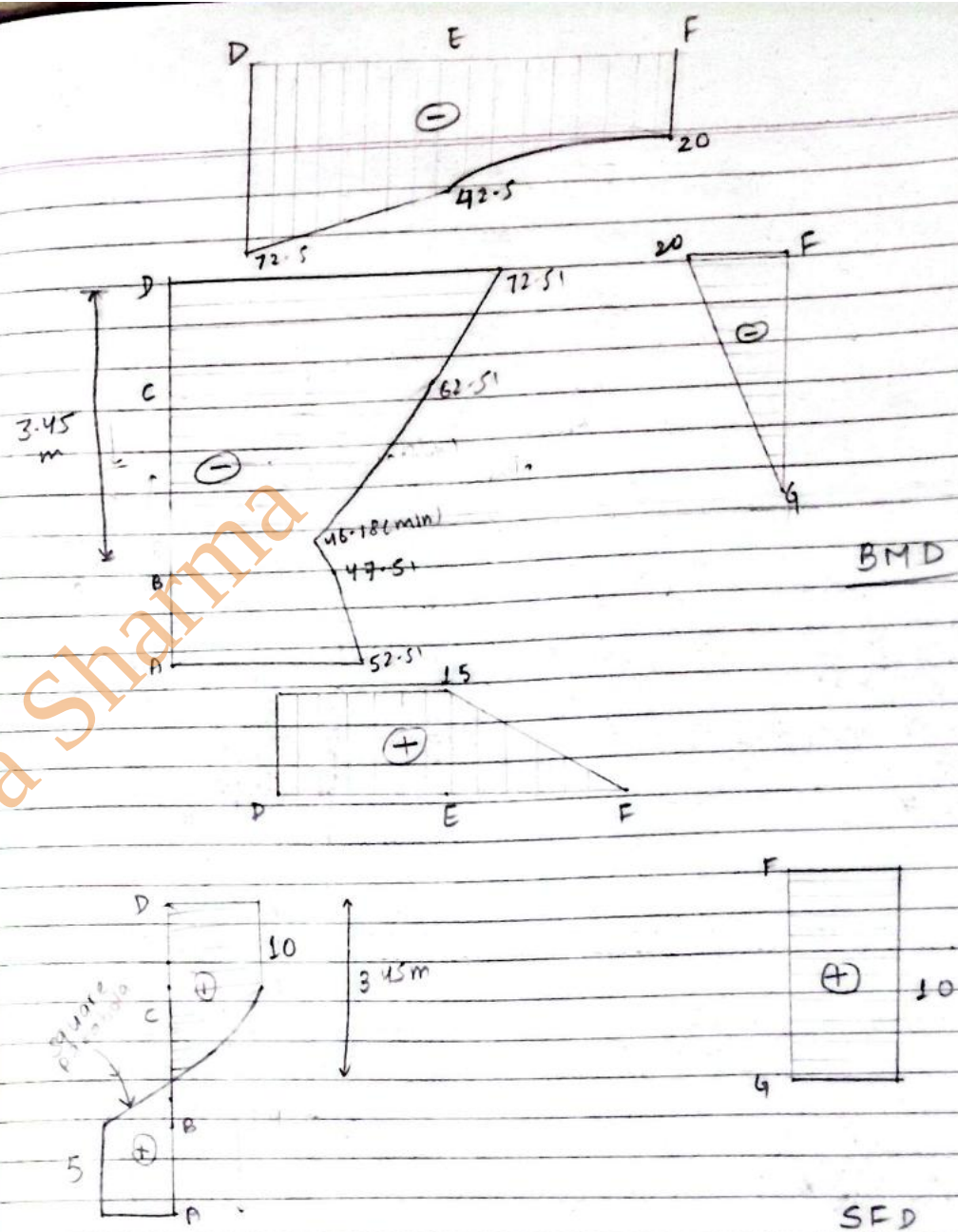
$$V_F = 15 - 15 = 0 \text{ kN}$$
$$(M_F)_R = -20 + 20 = 0 \text{ KNm}$$

R → L (9 +ve)
(↓ +ve)

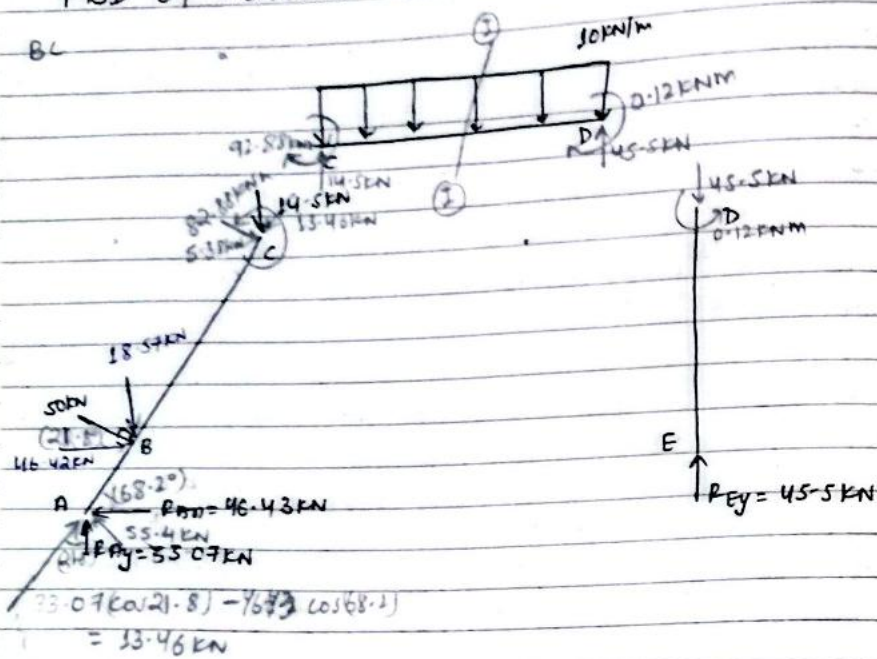


~~Ques 2~~ $V_0 = 5x$

$$M_n = -5n \cdot \frac{n}{2} - 20 = -2.5n^2 - 20$$

$$(V_4)_R = 10 - 10 = 0 \text{ kN}$$
$$M_G = -20 + 10 \times 2 = 0 \text{ kNm.}$$
$$F_{Fg} = 0 \text{ kN}$$


FBD of member of Frame with All reaction:



Now, For member AC

Axial Force:

$$F_{AC} = -13.46 \text{ kN (compressive)}$$

Shear Force Calculation:

Shear force just left of A: $(V_A)_L = 0 \text{ kN}$

$$(V_A)_R = 55.4 \text{ kN}$$

$$(V_B)_L = 55.4 \text{ kN}$$

$$(V_B)_R = 55.4 - 50 = 5.4 \text{ kN}$$

$$(V_C)_L = 5.4 \text{ kN}$$

$$(V_C)_R = 5.4 - 5.38 = 0.02 \text{ kN} \approx 0 \text{ kN}$$

Bending moment calculation:

Bending moment at A: $M_A = 0 \text{ kNm}$

$$M_B = 55.4 \times (AB = 1.08) = 59.832 \text{ kNm}$$

$$(M_C)_L = 55.4 \times (1.08) - 50 \times (BC = 4.31)$$

$$= 83.1 \text{ kNm}$$

$$(M_C)_R = 83.1 - 82.88 \approx 0 \text{ kNm}$$

For member CD

Axial Force: $F_{CD} = 0 \text{ kN}$

Shear force calculation:

$$(V_C)_L = 0 \text{ kN}$$

$$(V_C)_R = 14.5 \text{ kN}$$

$$(V_D)_L = 14.5 \text{ kN} - 10 \times 6 = -45.5 \text{ kN}$$

$$(V_D)_R = -45.5 + 45.5 = 0 \text{ kN}$$

Bending moment calculation:

$$(M_C)_L = 0 \text{ kNm}$$

$$(M_C)_R = 92.88 \text{ kNm}$$

$$(M_D)_L = 92.88 + 14.5 \times 6 - 10 \times 6 \times 3$$

$$= -0.12 \text{ kNm}$$

$$(M_D)_R = -0.12 + 0.12 = 0 \text{ kNm}$$

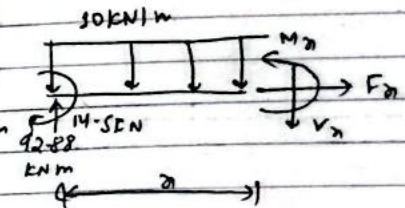
For section I-I

$$0 \leq x \leq 6$$

$$V_x = 14.5 - 10x \quad \text{For } V_x = 0 \Rightarrow x = 1.45 \text{ m}$$

$$M_x = 92.88 - 10x \times \frac{x}{2} + 14.5x$$

$$M_x = 92.88 - 5x^2 + 14.5x \quad \text{At } x = 1.45 \text{ m}, \quad M_x = 82.3675 \text{ kNm} \approx 103.39 \text{ kNm}$$



For member DE

Axial force: $F_{DE} = -45.5 \text{ kN (compressive)}$

Shear force: 0 kN

Bending moment $\approx 0 \text{ kNm}$

SFD BMD

(i) $\frac{dM}{dx} = V$ (shear force) { slope of BM is SF }

(ii) $\frac{dV}{dx} = LI$ { Load intensity }

(iii) $\uparrow$ point load $\Rightarrow$ SF jumps

(iv) $\downarrow$ point load $\Rightarrow$ SF falls

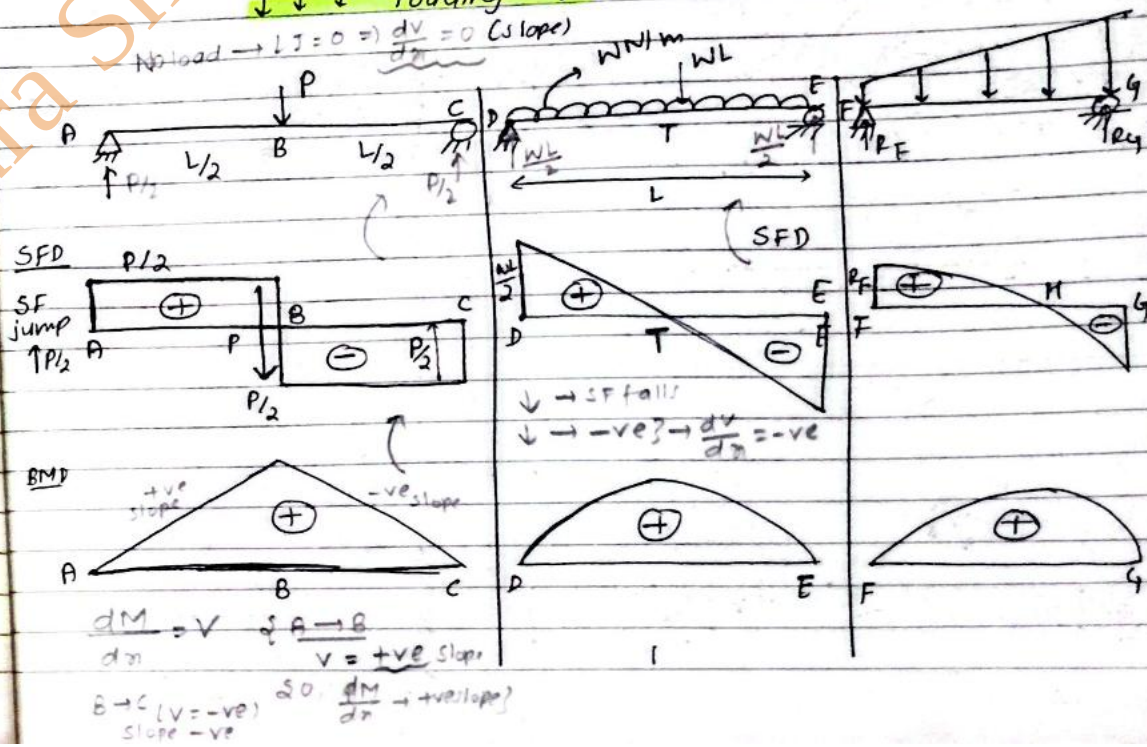
(v) $\curvearrowright$ Moment $\Rightarrow$ BMD jump

$\curvearrowleft$ Moment $\Rightarrow$ BMD falls

(vi) Upward load $\uparrow \uparrow \uparrow$ +ve

$\downarrow \downarrow \downarrow$ loading -ve

No load $\rightarrow LI = 0 \Rightarrow \frac{dV}{dx} = 0$ (slope)



AFD

SFD

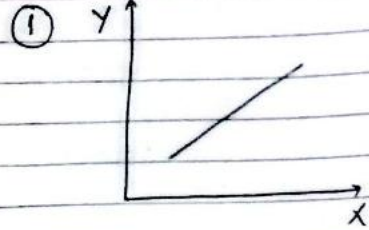
BMD

BMD

$\frac{dM}{dx} = V$ $\therefore$ A $\rightarrow$ B $V = +ve$ Slope $\therefore$ $\frac{dM}{dx} = +ve$ Slope

B $\rightarrow$ C ($V = -ve$) $\therefore$ $\frac{dM}{dx} = -ve$ Slope

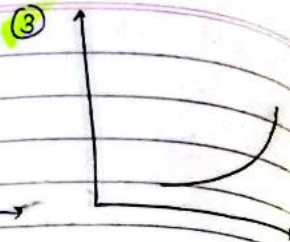
Graphs



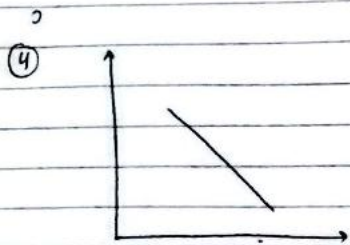
+ve & constant



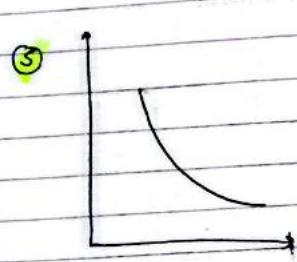
+ve & Mag. ↓



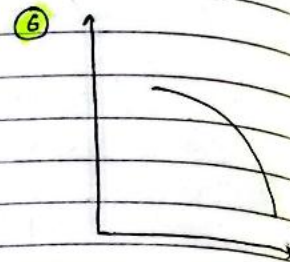
+ve & Mag. ↑



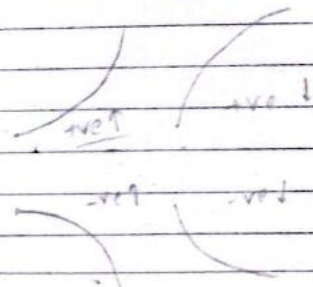
-ve & constant



-ve & Mag. ↓



-ve & Mag. ↑



Truss

• A pin jointed frame structural member is called truss.

• Assumptions of ideal truss:-

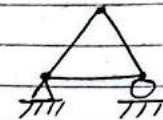
- (i) It is a pin jointed structural member i.e. it carries only axial force (since it is pin jointed)
- (ii) Loads are applied only at the joints.
- (iii) Self-weight of the member is neglected
- (iv) The centroidal axis of the member passes through the joint.
- (v) Truss is statically determinate

• Primary Truss (vi) Members connected by smooth frictionless pins

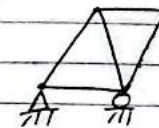
↳ Primary Truss is a simple triangle.

i.e. The basic configuration of truss is a simple triangle with three member and three joints (nodes)

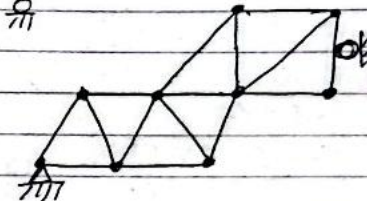
- 3 members
- 3 joints (nodes)



For further expansion of truss, two members and one joint is added.



Further,



Load transfer axially $\rightarrow$ More members req than a perfect truss
 Internal Indeterminacy $\rightarrow$ related to Member & their configuration

Determine indeterminacy of given Truss.

$\rightarrow m = 13$

$j = 8$

$r = 3$

Internal indeterminacy

$= m - 2j + 3$

$= 13 - 2 \times 8 + 3$

$= 0$

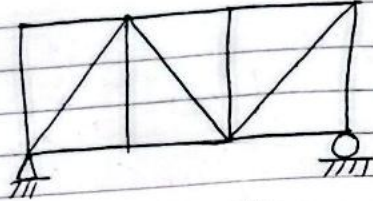


fig. (1) { Perfect Truss }

External indeterminacy $= r - 3 = 3 - 3 = 0$

Total indeterminacy $= m + r - 2j = 0$ { or internal + external }
 (statically indeterminate & stable)

$\rightarrow m = 14$

$j = 8$

$r = 3$

Int indeterminacy $= m - 2j + 3$
 (related to member) $= 14 - 2 \times 8 + 3$
 $= 1$

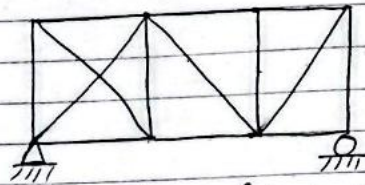


fig. (2) { Redundant Truss }

External indeterminacy $= r - 3 = 3 - 3 = 0$

Total indeterminacy $= \text{External SI} + \text{Internal SI} = 1$ (Indeterminate)

$\rightarrow m = 12$

$j = 8$

$r = 3$

External SI $= r - 3 = 3 - 3 = 0$

Internal SI $= m - 2j + 3 = 12 - 2 \times 8 + 3$
 (1 member less than statically determinate) $= -1$

Total Indeterminacy $= \text{External SI} + \text{Internal SI}$

$= -1$ (Unstable)

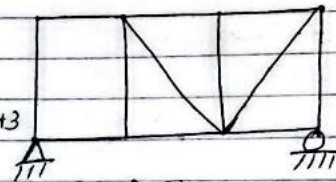


fig. (3)

(Imperfect Truss or deficient truss)



$\rightarrow$ 2 member truss square



$r - 3$

4 member truss square
 $m - 2j + 3$

$\rightarrow m = 13$

$j = 8$

$r = 4$

Internal SI $= m - 2j + 3 = 13 - 2 \times 8 + 3 = 0$

External SI $= r - 3 = 4 - 3 = 1$

Total SI $= \text{External SI} + \text{Internal SI}$

$= 1 + 0$

$= 1$ (Indeterminate)

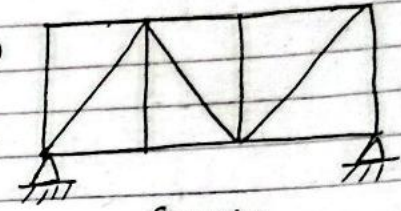


fig. (4)

$\rightarrow m = 14$

$j = 8$

$r = 4$

Internal SI $= m - 2j + 3 = 1$

External SI $= r - 3 = 4 - 3 = 1$

TSI $= \text{Internal SI} + \text{External SI}$

$= 1 + 1$

$= 2$ (Indeterminate)

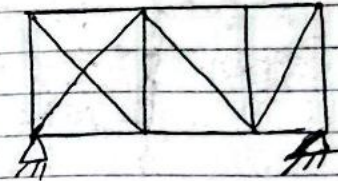


fig. (5)

$\rightarrow m = 12$

$j = 8$

$r = 4$

Internal SI $= m - 2j + 3 = 12 - 2 \times 8 + 3$
 $= -1$

External SI $= r - 3 = 4 - 3 = 1$

TSI $= \text{Internal SI} + \text{External SI}$

$= -1 + 1$

$= 0$ (Statically indeterminate & stable)
 Basic configuration of truss unit not fulfilled, the truss is not stable

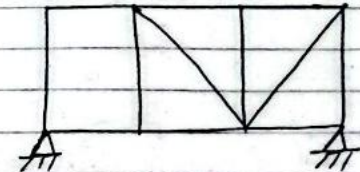


fig. (6)

Support reaction are calculated considering eqm of whole structure $\sum F_x = 0, \sum F_y = 0, \sum M = 0$

Analysis of Truss

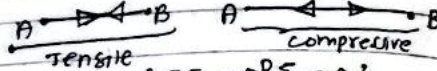
Basically, there are two methods:

- (1) Joint Method
- (2) Section Method

(1) Joint Method

- (A) (i) Select each joint
- (ii) Draw FBD of joint

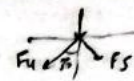
Away from joint $\rightarrow$ Tension (+ve)
Towards the joint $\rightarrow$ Compressive (-ve)



- (iii) Use equations of equilibrium $\sum F_x = 0$ & $\sum F_y = 0$
- (B) The maximum number of unknowns in a joint is 2.
- (C) It is suitable when force in all members is required.

(2) Section Method

- (A) (i) Taking one part of the section
- (ii) Drawing FBD of section
- (iii) Use equations of equilibrium $\sum F_x = 0, \sum F_y = 0$ & $\sum M = 0$
- (B) Generally, the maximum number of unknowns in this method is 3.
- (C) Suitable when force in some members is required.



$$2 F_y \cos 30^\circ = 4$$

(Q)

Soln.

$$\sum F_x = 0 (\rightarrow +ve)$$

$$\text{or } R_{Ax} = 0$$

$$\sum M_A = 0 (\uparrow +ve)$$

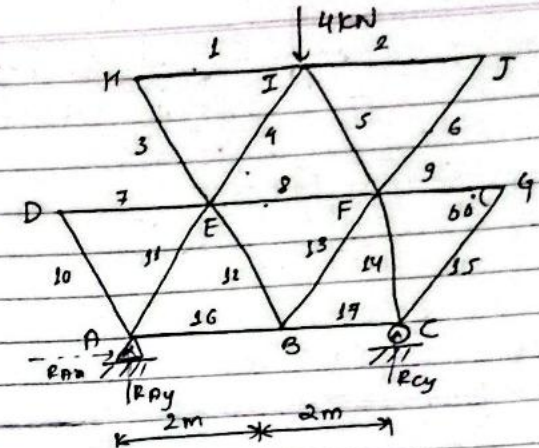
$$\text{or } 4 \times 2 - R_{Cy} \times 4 = 0$$

$$\text{or } R_{Cy} = 2 \text{ kN}$$

$$\sum F_y = 0 (\uparrow +ve)$$

$$\text{or } R_{Ay} + R_{Cy} - 4 = 0$$

$$\text{or } R_{Ay} = 2 \text{ kN}$$



In joint H,

$$\sum F_x = 0 (\rightarrow +ve)$$

$$\Rightarrow F_1 = 0$$

$$\sum F_y = 0 (\uparrow +ve)$$

$$-F_3 \cos 60^\circ = 0$$

$$\Rightarrow F_3 = 0$$

Joint J

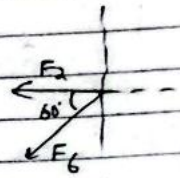
$$\sum F_x = 0 (\rightarrow +ve)$$

$$\Rightarrow F_2 = 0$$

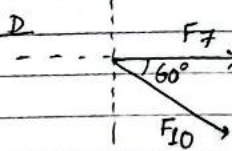
$$\sum F_y = 0 (\downarrow +ve)$$

$$F_6 \sin 60^\circ = 0$$

$$\Rightarrow F_6 = 0$$



For joint D



$$\sum F_y = 0 (\downarrow +ve)$$

$$\text{or } F_{10} \sin 60^\circ = 0$$

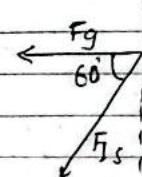
$$\Rightarrow F_{10} = 0 \text{ kN}$$

$$\sum F_x = 0 (\rightarrow +ve)$$

$$\text{or } F_7 + F_{10} \cos 60^\circ = 0$$

$$\text{or } F_7 = 0 \text{ kN}$$

For joint G,



$$\sum F_y = 0 (\uparrow +ve)$$

$$\text{or } F_{15} \sin 60^\circ = 0 \Rightarrow F_{15} = 0 \text{ kN}$$

$$\sum F_x = 0 (\rightarrow +ve)$$

$$-F_9 - F_{15} \cos 60^\circ = 0$$

$$\text{or } F_9 = 0 \text{ kN}$$

W Away from joint $\rightarrow +ve \rightarrow$ Tensile
Toward the joint $\rightarrow -ve \rightarrow$ Compressive

In Joint I

$$\sum F_x = 0 (\rightarrow +ve)$$

$$-F_4 \sin 30^\circ + F_5 \sin 30^\circ = 0$$

$$\Rightarrow F_4 = F_5$$



$$\sum F_y = 0 (\downarrow +ve)$$

$$4 + F_4 \cos 30^\circ + F_5 \cos 30^\circ = 0$$

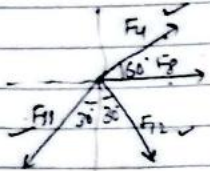
$$\text{or } F_4 = F_5 = \frac{-4}{2 \cos 30^\circ} = -2.31 \text{ kN}$$

(Compressive)

In Joint E

$$\sum F_y = 0 (\uparrow +ve)$$

$$\text{or } F_4 \sin 60^\circ - F_1 \cos 30^\circ - F_2 \cos 30^\circ = 0$$



$$\text{or } F_4 = F_5 = \frac{F_1 \sin 60^\circ - F_2 \cos 30^\circ}{\cos 30^\circ}$$

$$F_2 = 1.155 \text{ kN (Tensile)}$$

$$\sum F_x = 0 (\rightarrow +ve)$$

$$F_8 + F_2 \sin 30^\circ - F_1 \sin 30^\circ + F_4 \cos 60^\circ = 0$$

$$\text{or } F_8 = -1.155 \text{ kN (Compressive)}$$

In Joint A

$$\sum F_x = 0 (\rightarrow +ve)$$

$$F_{16} + F_1 \cos 60^\circ = 0$$

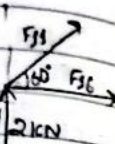
$$\therefore F_{16} = -2.31 \text{ kN (Compressive)}$$

$$\sum F_y = 0 (\uparrow +ve)$$

$$2 + F_{11} \sin 60^\circ = 0$$

$$\text{or } F_{11} = -2.31 \text{ kN}$$

$$F_{11} = 2.31 \text{ kN (Compressive)}$$



For Joint F

$$\sum F_x = 0 (\rightarrow +ve)$$

$$\text{or } -F_8 - F_5 \cos 60^\circ - F_{13} \cos 60^\circ + F_{14} \cos 60^\circ = 0$$

$$\text{or } 0.5 F_{14} - 0.5 F_{13} - 0.5 F_5 = 0$$

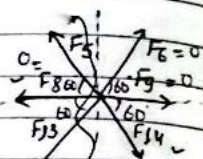
$$\text{or } 0.5 F_{13} + F_5 = 1.155 \text{ (1)}$$

$$\sum F_y = 0 (\uparrow +ve)$$

$$\text{or } F_5 \sin 60^\circ - F_{13} \sin 60^\circ - F_{14} \sin 60^\circ = 0$$

$$\text{or } -0.866 F_{13} + 0.866 F_5 = 2 \text{ (2)}$$

$$\text{Solving (1) & (2)}$$



For Joint C

$$\sum F_y = 0 (\uparrow +ve)$$

$$\text{or } F_{14} \sin 60^\circ + 2 = 0$$

$$\text{or } F_{14} = -2.31 \text{ kN}$$

$$F_{14} = 2.31 \text{ kN}$$

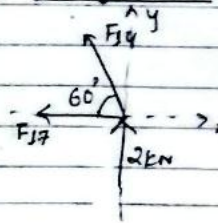
(Compressive)

$$\sum F_x = 0 (\rightarrow +ve)$$

$$\text{or } -F_{17} - F_{14} \cos 60^\circ = 0$$

$$\text{or } F_{17} = 1.155 \text{ kN}$$

$$(Tensile)$$



For Joint F

$$\sum F_y = 0 (\uparrow +ve)$$

$$F_5 \sin 60^\circ - F_{13} \sin 60^\circ - F_{14} \sin 60^\circ = 0$$

$$\text{or } 0.866 F_5 - 0.866 F_{13} = 2 \text{ (1)}$$

$$\text{For Joint F } \sum F_x = 0 (\rightarrow +ve)$$

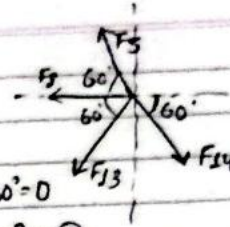
$$-F_8 - F_5 \cos 60^\circ - F_{13} \cos 60^\circ + F_{14} \cos 60^\circ = 0$$

$$\text{or } -0.5 F_5 - 0.5 F_{13} = -1.155 \text{ (2)}$$

$$\text{Solving (1) & (2)}$$

$$F_5 = 2.31 \text{ kN (Tensile)}$$

$$F_{13} \approx 0 \text{ kN}$$



For Joint B

$$\sum F_y = 0 (\uparrow +ve)$$

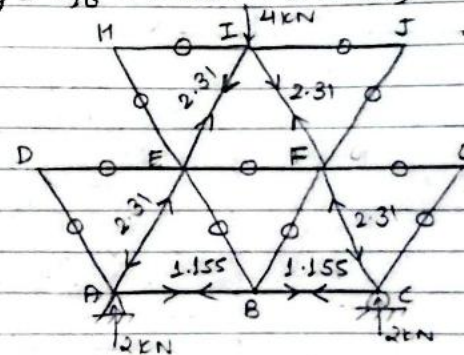
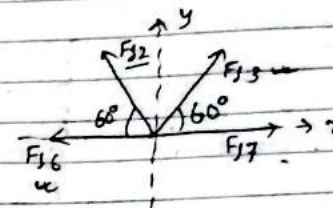
$$\text{or } F_{12} \sin 60^\circ + F_{13} \sin 60^\circ = 0$$

$$\Rightarrow F_{12} = 0 \text{ kN}$$

$$\sum F_x = 0 (\rightarrow +ve)$$

$$-F_{16} - F_{12} \cos 60^\circ + F_{13} \cos 60^\circ + F_{17} = 0$$

$$\text{or } F_{17} = F_{16} = 1.155 \text{ kN (Tensile)}$$



Zero force Members

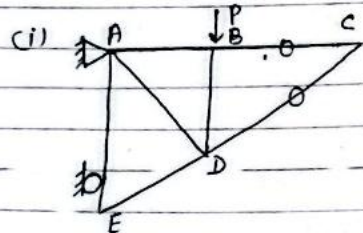
↳ The member which carries no force is called zero force members. There are two general cases in which zero force member arises in truss:-

✓ (i) When two members meet at a joint (form a joint) non-collinear

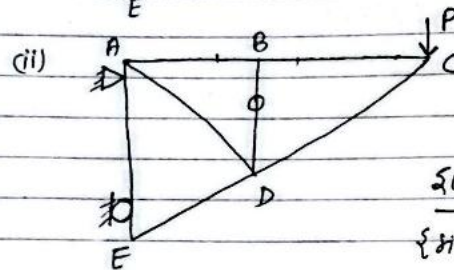
& there is no force or support reaction at that joint then the members are zero force members.

(ii) When three members form a joint in which two of the members are collinear and there is no force or support reactions at that joint then, the third member is zero force member. And, the two members carry equal and opposite force. (with ref. to that joint)

(Both in tension or Both in compression)



$$F_{BC} = 0 \quad F_{DC} = 0 \quad \left\{ \begin{array}{l} \text{cond} = \text{cl} \end{array} \right\}$$



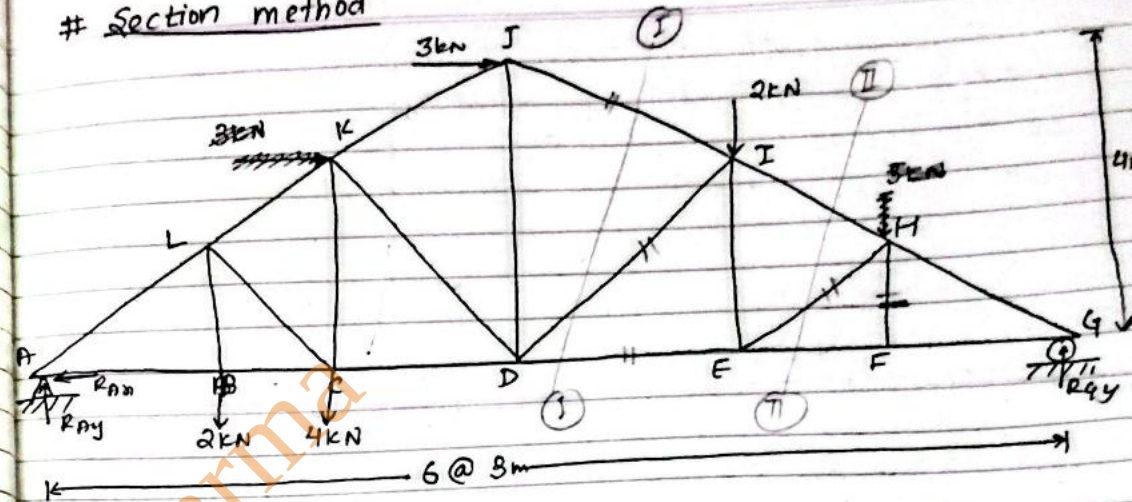
$$F_{BD} = 0 \quad \left\{ \begin{array}{l} \text{cond} = \text{cl} \end{array} \right\}$$

Now, ↓

$$\left\{ \begin{array}{l} \text{So, } F_{AD} = 0 \quad \left\{ \begin{array}{l} \text{cond} = \text{cl} \end{array} \right\} \\ \text{Again, } F_{BD} = 0, \text{ three members are AD, ED \& DC} \end{array} \right\}$$

$$\& \quad F_{DE} = F_{DC}$$

Section method



$$\sum F_x = 0 \quad (\rightarrow +ve)$$

$$\Rightarrow [R_{Ax} = 3 \text{ kN}]$$

$$\sum M_D = 0 \quad (\text{Clockwise} +ve)$$

$$3 \times 4 - R_{Gy} \times 18 + 2 \times 12 + 3 \times 15 + 2 \times 3 + 4 \times 6 = 0$$

$$[R_{Gy} = 3.67 \text{ kN}]$$

$$\sum F_y = 0 \quad (\uparrow +ve)$$

$$-2 - 2 - 4 + R_{Ay} + R_{Gy} = 0$$

$$\Rightarrow [R_{Ay} = 4.33 \text{ kN}]$$

In section I-I

For finding unknown reactions

Method-I

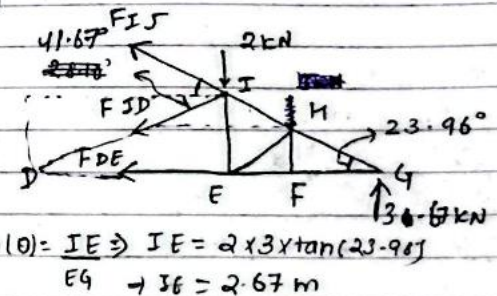
$$\sum M_G = 0 \quad (\text{Clockwise} +ve)$$

$$0.1. F_{IJ} \times 10 + F_{DE} \times 0 - 2 \times 6$$

$$- F_{ID} \cos(41.67^\circ) \times 2.67$$

$$- F_{ID} \sin(41.67^\circ) \times 6 = 0$$

$$\Rightarrow F_{ID} = 2.22 \text{ kN} \quad \Rightarrow [F_{ID} = 2.22 \text{ kN (C)}]$$



$$\tan(\theta) = \frac{IE}{IE} \Rightarrow IE = 2 \times 3 \times \tan(23.96^\circ)$$

$$EG \rightarrow 36 = 2.67 \text{ m}$$

$$\Sigma F_y = 0 \quad (\uparrow +ve)$$

$$\text{or, } 3.67 - 2 + F_{JJ} \sin(23.96^\circ) - F_{JD} \sin(41.67^\circ) = 0$$

$$\text{or, } F_{JJ} = \frac{7.39 \text{ kN}}{\sin(23.96^\circ)}$$

$$[F_{JJ} = 12.97 \text{ kN(C)}]$$

$$\Sigma F_x = 0 \quad (\rightarrow +ve)$$

$$-F_{JJ} \cos(23.96^\circ) - F_{JD} \cos(41.67^\circ) - F_{DE} = 0$$

$$\text{or, } [F_{DE} = 13.28 \text{ kN(T)}]$$

$$\text{OR } \Sigma M_J = 0 \quad (\uparrow +ve) \quad \{\text{For } F_{DE}\}$$

$$\text{or, } F_{DE} \times 2.67 - 3.67 \times 6 = 0$$

$$\text{or, } [F_{DE} = 13.28 \text{ kN(T)}]$$

Also,

we can find F_{JJ} by taking moment about D point (external of considered section) equal to zero as well

$$\Sigma M_D = 0 \quad (\uparrow +ve) \quad \{F_{JD} \& F_{DE} \text{ gives no moment}\}$$

$$\text{or, } -F_{JJ} \sin(23.96^\circ) \times 3 - F_{JJ} \cos(23.96^\circ) \times 2.67 + 2 \times 3 - 6.17 \times 9 = 0$$

$$\text{or, } F_{JJ} = \frac{2 \times 3 - 6.17 \times 9}{\sin(23.96^\circ) \times 3 + \cos(23.96^\circ) \times 2.67}$$

$$F_{JJ} = -7.39 \text{ kN}$$

$$\therefore [F_{JJ} = 7.39 \text{ kN(C)}]$$

We can find all unknowns by taking moment about D (for F_{JJ}), about G (for F_{JD}) and about J (for F_{DE}) as well.

From every way, reaction calculated will be same

$$F_{JJ} = 7.39 \text{ kN(C)}$$

$$F_{JD} = 2.25 \text{ kN(C)}$$

$$F_{DE} = 13.28 \text{ kN(T)}$$

We know,

For section I - II

($F_{HF} = 0 \text{ kN}$) $\Rightarrow$ sin, Three

members HF, EF & FG

form a joint F such that

two members FE & FG are

collinear and there is no

support rxn or force at joint F

then, the force in member

HF (third member) is zero i.e.

HF is zero force member

Here,

we just need F_{HE} thus, easiest way is to

$$\Sigma M_G = 0 \quad (\uparrow +ve) \quad \{\text{Will give } F_{HE} \text{ only}\}$$

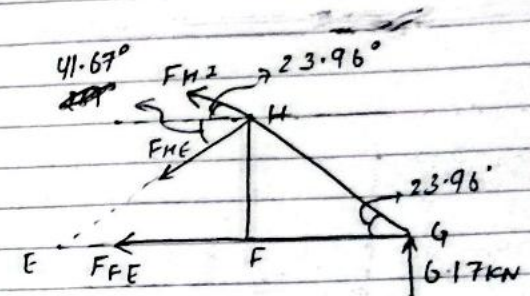
$$\text{or, } -F_{HE} \sin(41.67^\circ) \times 3 - F_{HE} \cos(41.67^\circ) \times 1.33 = 0$$

$$\therefore [F_{HE} = 0 \text{ kN}]$$

OR' Since, $F_{HF} = 0 \text{ kN}$

Joint H can be considered as joint forming three members HI, HF & HG. Thus, two members HI & HG are collinear & no force acts on point/joint/node H thus, third member HE is zero force member

$$\text{i.e. } [F_{HF} = 0 \text{ kN}]$$



$$\tan(23.96^\circ) = \frac{MF}{F_G} \Rightarrow MF = 1.33 \text{ m}$$

S.N	Member	Force (nature)
1	ST	7.39 kN (C)
2	ID	2 kN (C)
3	SE	8.25 kN (T)
4	HF	0 kN (Zero force member)
5	HE	0 kN (Zero force member)

Surendra Sharma

② Soln:

A.T.P.

Point of contraflexure A
is at the midpoint
of beam

i.e. $(BM)_{mid} = 0$

i.e. $(BM)_E = 0$

Analyzing beam from right to left
($\curvearrowright +ve$)

$$160 \times 5 \times 2.5 - R_{Dy}(5-z) = 0 \quad \text{--- (I)}$$

$$\text{or } R_{Dy}z = 5R_{Dy} - 2000 \quad \text{--- (II)}$$

Now, for stable eqn,

$$\sum F_x = 0 \quad (\rightarrow +ve)$$

$$\Rightarrow [R_{Cx} = 0 \text{ N}]$$

$$\sum M_C = 0 \quad (\curvearrowright +ve)$$

$$\text{or } -400 \times 1 + 160 \times 1 \times 0.5 - R_{Dy}(9-z) + 160 \times 9 \times 4.5 = 0$$

$$\text{or } 6000 - 9R_{Dy} + R_{Dy}z = 0$$

$$\text{or } 6000 - 9R_{Dy} + 5R_{Dy} - 2000 = 0 \quad \{\text{from (II)}\}$$

$$\text{or } -4R_{Dy} = -4000$$

$$\text{or } [R_{Dy} = 1000 \text{ N} (\uparrow)]$$

$$\sum F_y = 0 \quad (\uparrow +ve)$$

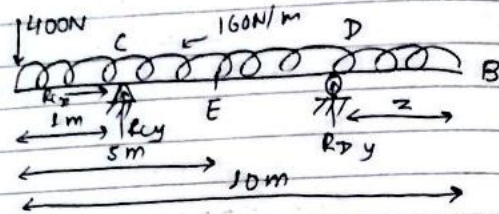
$$-400 - 160 \times 10 + R_{Cy} + R_{Dy} = 0$$

$$\text{or } [R_{Cy} = 1000 \text{ N}]$$

Thus, from (II)

$$z = \frac{5 \times 1000 - 2000}{1000} = 3 \text{ m}$$

$$[z = 3 \text{ m}]$$



Axial force calculation

$$[F_{AB} = 0]$$

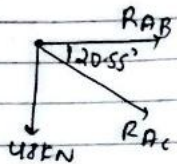
Shear force calculation

S.F. just left of A (F)

③

Soln.

For joint A



$$R_{AC} = \frac{-48}{\sin 20.55} = -136.74 \text{ kN}$$

$$[R_{AC} = 136.74 \text{ kN (C)}]$$

$$R_{AB} = -R_{AC} \cos(20.55)$$

$$[R_{AB} = 128.04 \text{ (T)}]$$

$$[R_{CE} = R_{AC} = 128.04 \text{ (T)}]$$

$$R_{BD} = R_{AB} \cdot [R_{CE} = R_{AC} = 136.74 \text{ kN (C)}]$$

$$[R_{BD} = R_{AB} = 128.04 \text{ (T)}]$$

$$[R_{EG} = R_{CE} = 136.74 \text{ kN (C)}]$$

$$[R_{DF} = R_{DB} = 128.04 \text{ (T)}]$$

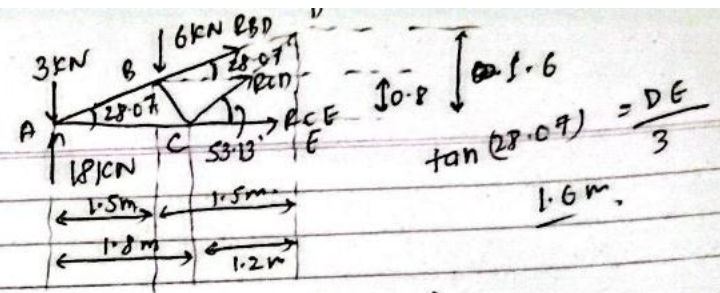
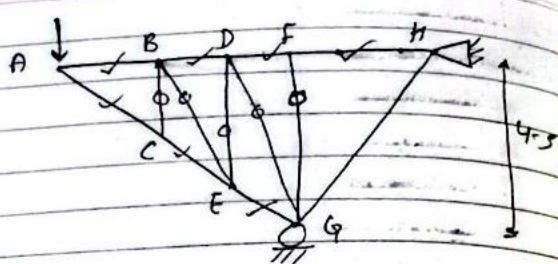
$$[R_{FH} = R_{FD} = 128.04 \text{ (T)}]$$

$$R_{GH} = \frac{R_{GE} \cos(20.55)}{\cos 48.37} = \frac{-136.74 \text{ kN}}{\cos 48.37}$$

$$= -192.7$$

$$[R_{GH} = 192.7 \text{ kN (C)}]$$

4 @ 4m = 16m



$$\sum M_A = 0 \text{ (12ve)}$$

$$\text{or } 6 \times 1.5 - R_{CD} \sin(53.13) \times 1.8 = 0$$

$$\text{or } [R_{CD} = 6.25 \text{ kN (T)}]$$

$$\sum F_y = 0 \text{ (12ve)}$$

$$\text{or } 18 - 3 - 6 + R_{CD} \sin(53.13) + R_{BD} \sin(28.07) = 0$$

$$\text{or } R_{BD} = \frac{-14}{\sin(28.07)}$$

$$[R_{BD} = 28.75 \text{ kN (C)}]$$

$$R_{CE} = -R_{BC} \cos(28.07) - R_{CD} \cos(53.13)$$

$$[R_{CE} = 22.5 \text{ kN (T)}]$$

velocity tangent to path.

$$\vec{v} = v \hat{e}_t + 0 \hat{e}_n$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d(v \hat{e}_t)}{dt}$$

$$= \dot{v} \hat{e}_t + v \frac{d\hat{e}_t}{dt}$$

$$= \dot{v} \hat{e}_t + v \frac{d\hat{e}_t}{d\theta} \cdot \frac{d\theta}{ds} \cdot \frac{ds}{dt}$$

$$\frac{d\hat{e}_t}{d\theta} \lim_{\Delta\theta \rightarrow 0} \frac{\Delta\hat{e}_t}{\Delta\theta} = \lim_{\Delta\theta \rightarrow 0} \frac{(\Delta\theta \times 1) \hat{e}_n}{\Delta\theta} = \hat{e}_n$$

$$\frac{d\theta}{ds} = \lim_{\Delta\theta \rightarrow 0} \frac{\Delta\theta}{\Delta s} = \frac{\Delta s}{\Delta\theta} = \frac{1}{\rho}$$

$$= \dot{v} \hat{e}_t + v \hat{e}_n \frac{1}{\rho} \times \rho$$

$$\vec{a} = \dot{v} \hat{e}_t + \frac{v^2}{\rho} \hat{e}_n$$

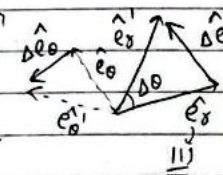
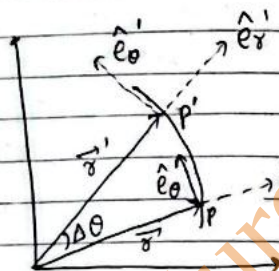
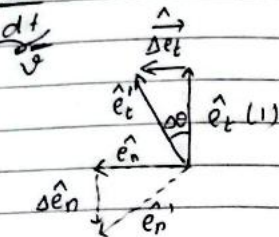
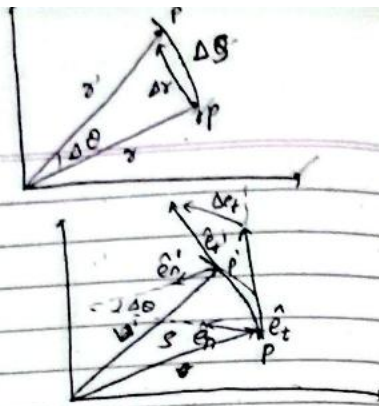
$$\frac{1}{\rho} = \frac{a_n}{v^2}$$

$$\vec{r} = r \hat{e}_r$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r} \hat{e}_r + r \frac{d\hat{e}_r}{dt} \cdot \frac{d\theta}{dt}$$

$$\frac{d\hat{e}_r}{d\theta} = \lim_{\Delta\theta \rightarrow 0} \frac{\Delta\hat{e}_r}{\Delta\theta} = \frac{(\Delta\theta \times 1) \hat{e}_\theta}{\Delta\theta} = \hat{e}_\theta$$

$$\vec{v} = \dot{r} \hat{e}_r + r \dot{\theta} \hat{e}_\theta$$



Q Solution:-

calculation of unknown reactions,

$$\sum M_A = 0 (\uparrow +ve)$$

$$\text{on } 30 \times 4 \times 5 - 50 \times 2$$

$$- R_{Ex} \times 7 = 0$$

$$\text{or } [R_{Ex} = 71.43 \text{ kN} (\rightarrow)]$$

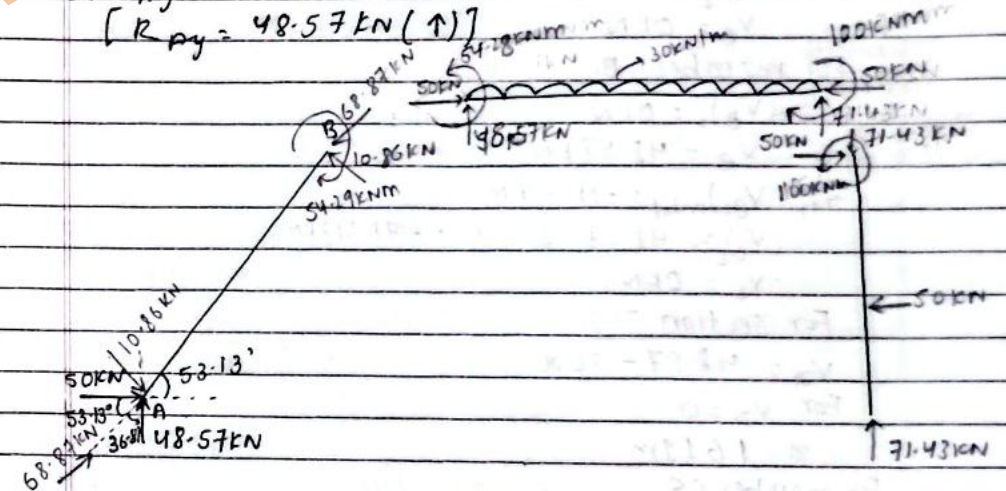
$$\sum F_x = 0 (\rightarrow +ve)$$

$$[R_{Ax} = 50 \text{ kN} (\rightarrow)]$$

$$\sum F_y = 0 (\uparrow +ve)$$

$$\text{on } R_{Ay} = 30 \times 4 - 71.43$$

$$[R_{Ay} = 48.57 \text{ kN} (\uparrow)]$$



Axial force calculation

For member AB

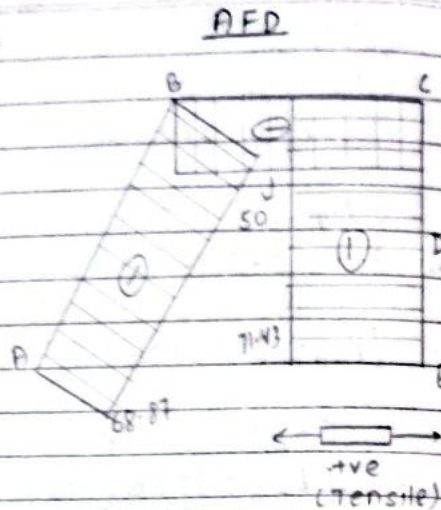
$$F_{AB} = -68.87 \text{ kN}$$

For member BC

$$F_{BC} = -50 \text{ kN}$$

For member CD

$$F_{CE} = -71.43 \text{ kN}$$



Shear force calculation

For member AB

Shear force just left of A $(V_A)_L = 0 \text{ kN}$

$$V_A = -10.86 \text{ kN}$$

$$(V_B)_L = -10.86 \text{ kN}$$

$$V_B = 0 \text{ kN}$$

For member BC

$$(V_B)_L = 0 \text{ kN}$$

$$V_B = 48.57 \text{ kN}$$

$$(V_C)_{\text{mid}} = -11.43 \text{ kN}$$

$$(V_C)_L = 48.57 - 30 \times 4 = -71.43 \text{ kN}$$

$$V_C = 0 \text{ kN}$$

For section J-I

$$V_J = 48.57 - 30x$$

For $V_J = 0$

$$x = 1.619 \text{ m}$$

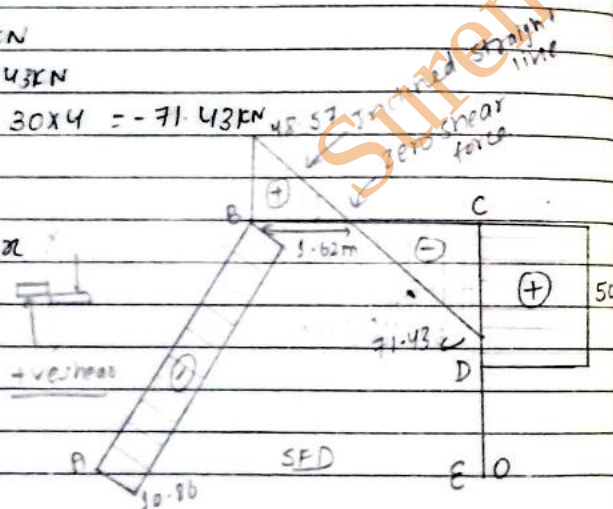
For member CE

$$(V_C)_L = 0 \text{ kN}$$

$$V_C = 50 \text{ kN}$$

$$(V_D)_L = 50 \text{ kN}$$

$$V_D = 0 \text{ kN} \quad V_E = 0 \text{ kN}$$



Bending moment calculation

Bending moment just left of A $(M_A)_L = 0 \text{ kNm}$

$$M_A = 0 \text{ kNm}$$

$$(M_B)_L = -54.29 \text{ kNm}$$

$$M_B = 0 \text{ kNm}$$

For member BC

$$(M_B)_L = 0 \text{ kNm}$$

$$M_B = -54.28 \text{ kNm}$$

$$(M_D)_L = -100 \text{ kNm}$$

$$M_D = 0 \text{ kNm}$$

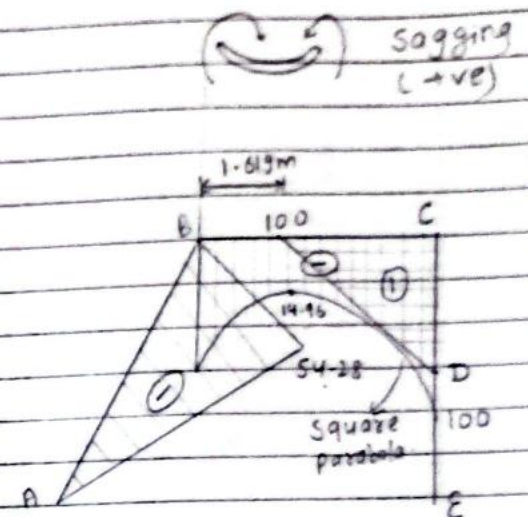
For member CE

$$(M_C)_L = 0 \text{ kNm}$$

$$M_C = -100 \text{ kNm}$$

$$M_D = 0 \text{ kNm}$$

$$M_E = 0 \text{ kNm}$$



For section J-I

$$M_J = -54.28 + 48.57x - 30x \times \frac{x}{2}$$

$$M_J = -54.28 + 48.57x - 15x^2$$

$$\text{At } x = 1.619 \text{ m}$$

$$M_J = -14.96 \text{ kNm}$$

BMD Bending Moment Diagram (B-M-D)

Two system of forces $\vec{F}_1, \vec{F}_2, \vec{F}_3, \dots$ and $\vec{F}_1', \vec{F}_2', \vec{F}_3', \dots$ which act on the same rigid body are equivalent if and only if the sum of forces & sum of moments about a given point O of the forces of two systems are respectively equal. i.e. $\sum \vec{F} = \sum \vec{F}'$ and $\sum \vec{M}_O = \sum \vec{M}'_O$ — (I)

→ second relation established w.r.t. only one point 'O'

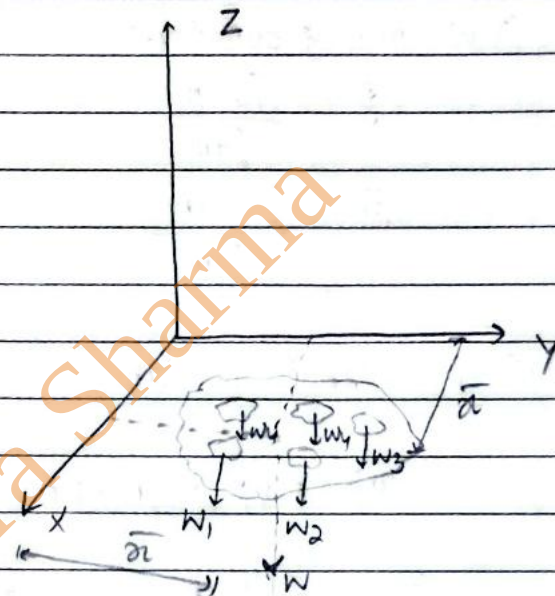
→ Two systems ~~of~~ can be reduced to same force-couple system at given pt. 'O'

If two system of vectors (forces) satisfy relation (I) when their resultants and their moment resultants about an arbitrary point 'O' are respectively equal, the two systems are said to be equipollent.

→ If two systems of forces acting on a rigid body are equipollent, they are also equivalent.

Centre of Gravity (CG)

→ The Centre of Gravity (CG) of a body is that point through which whole weight of the body acts irrespective of position of body i.e. The CG of a body is that point of the body through which system of parallel forces formed by weight of all the particles of the body passes for all position of body.

# Centroid:

→ It is the term used for figures without reference to weight of them. The plane figures like circle, triangle, square, quarter-circle, etc. have only areas but no mass. The centre of area of such figure is called centroid. It is also called centre of Geometry.

→ Consider,

$w_1, w_2, w_3, w_4, w_5, \dots$ → weight of elements with coordinates $(x_1, y_1), (x_2, y_2), (x_3, y_3), \dots$

Now,

Moment about X-axis (M_{x-axis}) = $W \times \bar{y} = w_1 y_1 + w_2 y_2 + w_3 y_3 + \dots$

$$\therefore \bar{y} = \frac{\sum w_i y_i}{W} = \frac{\sum_{i=1}^n w_i y_i}{\sum_{i=1}^n w_i}$$

Discrete structure: $\sum$ का उपयोग
 Continuous structure: $\int$ का उपयोग

Taking moment about y-axis
 $\bar{x} = \frac{\sum w_i x_i}{\sum w_i}$

When $h \rightarrow \infty$

$$\bar{y} = \frac{\int y dw}{\int dw}$$

$$\bar{x} = \frac{\int x dw}{\int dw}$$

① $W = mg$, g constant

$$\bar{y} = \frac{\int y dm}{\int dm}$$

$$\bar{x} = \frac{\int x dm}{\int dm}$$

② If slab is of uniform thickness then,

$$w_i = \rho V_i = \rho A_i t \quad \left\{ \begin{array}{l} \gamma = \frac{mg}{V} = \rho \times g \\ \text{Unit weight} \end{array} \right.$$

$$\therefore \bar{x} = \frac{\int x dA}{\int dA} = \frac{\sum x_i A_i}{\sum A_i} \quad \text{Moments of area about } (y) \perp x\text{-axis.}$$

$$\bar{y} = \frac{\int y dA}{\int dA} = \frac{\sum y_i A_i}{\sum A_i}$$

For continuous structure $\leftarrow$ For discrete structure

Axis of symmetry:-

(3) If any structure have same thickness & width then

$$\bar{x} = \frac{\sum x_i l_i}{\sum l_i} \quad \bar{y} = \frac{\sum y_i l_i}{\sum l_i}$$

Centroid of line
 (1) Quarter circle

	$\bar{x}$	$\bar{y}$	length
(1) Quarter circle	$\frac{2r}{\pi}$	$\frac{2r}{\pi}$	$\frac{\pi r}{2}$

Axis of symmetry

→ The line or axis or plane which divides the given line or area or volume into two identical equal parts is known as axis of symmetry.

→ In case of homogeneous body, the weight on one side of an axis of symmetry equals to that on the other side. And hence C.G. lies on axis of symmetry.

→ C.G. must lie on axis of symmetry if the area is symmetrical and the first moment of area about that axis must be zero

$$\text{i.e. } \bar{x} = \frac{\sum A_i x_i}{\sum A_i} = 0 \quad \text{or } \bar{y} = \frac{\sum A_i y_i}{\sum A_i} = 0$$

(If ref. axis is axis of symmetry)

Centroid of line
 (1) Quarter circle



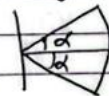
	$\bar{x}$	$\bar{y}$	length
(1) Quarter circle	$\frac{2r}{\pi}$	$\frac{2r}{\pi}$	$\frac{\pi r}{2}$

(2) Semi-circle



	$\bar{x}$	$\bar{y}$	length
(2) Semi-circle	0	$\frac{2r}{\pi}$	πr

(3) Arc of circle



	$\bar{x}$	$\bar{y}$	length
(3) Arc of circle	$\frac{r \sin \alpha}{\alpha}$	0	$2r\alpha$

Center of area

(1) Quarter circle



$$\bar{x} = \frac{4r}{3\pi} \quad \bar{y} = \frac{4r}{3\pi} \quad \text{Area} = \frac{\pi r^2}{4}$$

(2) Semi-circle



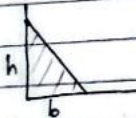
$$\bar{x} = 0 \quad \bar{y} = \frac{4r}{3\pi} \quad \text{Area} = \frac{\pi r^2}{2}$$

(3) Quarter Elliptical area



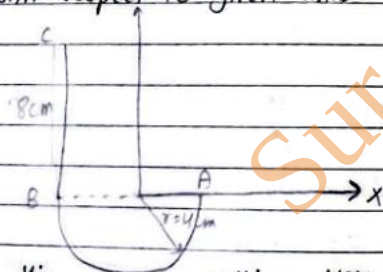
$$\bar{x} = \frac{4a}{3\pi} \quad \bar{y} = \frac{4b}{3\pi} \quad \text{Area} = \frac{\pi ab}{4}$$

(4) Triangle



$$\bar{x} = \frac{b}{3} \quad \bar{y} = \frac{h}{3} \quad \text{Area} = \frac{1}{2}bh$$

Q. A slender homogeneous wire of uniform cross-section is bent into a form as shown in figure. Determine the centroid of wire with respect to given axis.



Part	x_i	y_i	L_i	$x_i L_i$	$y_i L_i$
1. Semicircular	0	$-\frac{2 \times 4}{\pi}$	$\pi \times 4$	0	$-32/\pi$
2. Straight line	-4	4	8	-32	32

$$\sum x_i L_i = -4 \quad \sum y_i L_i = 1.45 \quad \sum L_i = 20.56 \quad \sum x_i L_i = -32 \quad \sum y_i L_i = 0$$

Now, $\bar{x} = \frac{\sum x_i L_i}{\sum L_i} = -1.56 \text{ cm}$ $\bar{y} = \frac{\sum y_i L_i}{\sum L_i} = 0 \text{ cm}$

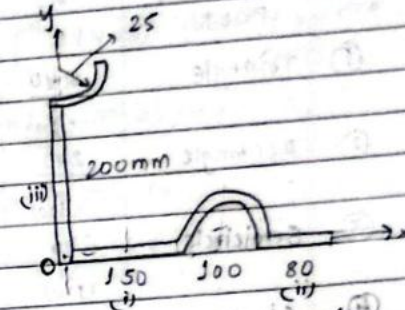
Q. Find $\bar{x}$ & $\bar{y}$.

Parts	x_i	y_i	L_i	$x_i L_i$	$y_i L_i$
(1) Line (i)	75	0	150	11250	0
(2) Semi-circle	200	81.83	157.08	31416	4999.86
(3) Line (ii)	290	0	80	23200	0
(4) Line (iii)	0	100	200	0	20000
(5) Quarter circle	15.91	215.91	39.27	624.76	8478.79

$$\sum L_i = 626.35$$

$$\bar{x} = \frac{\sum x_i L_i}{\sum L_i} = \frac{66496.76}{626.35} = 106.16 \text{ mm}$$

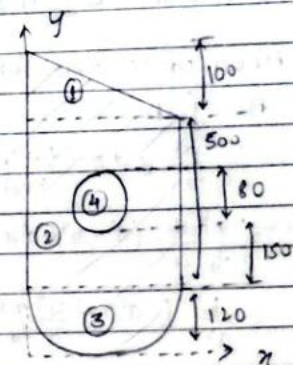
$$\bar{y} = \frac{\sum y_i L_i}{\sum L_i} = \frac{33478.65}{626.35} = 53.45 \text{ mm}$$



Length of semi-circle = $\frac{2\pi r}{2} = \pi r$
 Length of quarter circle = $\frac{\pi r}{2}$

Q. For the shaded plane area shown, determine the location of centroid. (Composite figure)

Component parts	x_i	y_i	A_i	$x_i A_i$	$y_i A_i$
(1) Triangle	$\frac{1}{3} \times 140$	$\frac{1}{3} \times 100$	$\frac{1}{2} \times 140 \times 100$		
	= 80	= 33.33			



Component Part	x_i	y_i	A_i	$\Sigma x_i A_i$	$\Sigma y_i A_i$
① Triangle	$\frac{240}{3}$ (80)	$\frac{120+500+100}{3}$ (653.33)	$\frac{1}{2} \times 240 \times 100$ (12000)	960000	7839360
② Rectangle	$\frac{240}{2}$ (120)	$\frac{120+500}{2}$ (370)	240×500 (120000)	14400000	44400000
③ Semicircle	$\frac{240}{2}$ (120)	$\frac{120-4 \times \frac{120}{3\pi}}{2}$ (69.07)	$\frac{\pi (120)^2}{2}$ (22619.47)	2713976.4	1562326.8
④ Circle	$\frac{240}{2}$ (120)	$\frac{120+150+80}{2}$ (310)	$-\pi (40)^2$ (-5026.55)	$\Sigma x_i A_i =$ 17470770.4	$\Sigma y_i A_i =$ 52244056.3

Now,

$$\bar{x} = \frac{\Sigma x_i A_i}{\Sigma A_i} = 116.79 \text{ cm}$$

$$\bar{y} = \frac{\Sigma y_i A_i}{\Sigma A_i} = 349.24 \text{ cm}$$

Thus,

$$\text{Centroid} = (116.79, 349.24)$$

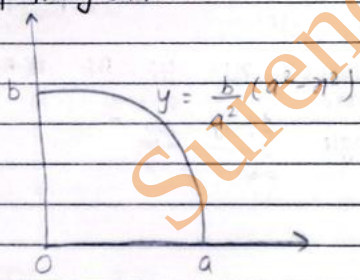
Method 4

Centroid location by integration

Q. Locate the centroid of the area under the curve as shown by method of integration

→ Area under the curve

$$\begin{aligned} A &= \int dA \\ &= \int_0^a y \, dx \quad (\text{taking vertical strip}) \\ &= \int_0^a \frac{b}{a^2} (a^2 - x^2) \, dx \\ &= \frac{b}{a^2} \left[a^2 x - \frac{x^3}{3} \right]_0^a \\ &= \frac{2}{3} ab \end{aligned}$$



Now,

$$\begin{aligned} \bar{x} &= \frac{\int x \, dA}{\int dA} = \frac{1}{A} \int x \, dA = \frac{3}{2ab} \int_0^a x (y \, dx) \quad \{\text{taking vertical strip}\} \\ &= \frac{3}{2ab} \int_0^a x \left(\frac{b}{a^2} (a^2 - x^2) \right) dx \\ &= \frac{3b}{2a^3 b} \int_0^a (a^2 x - x^3) dx \\ &= \frac{3}{2a^3} \left[\frac{a^2 x^2}{2} - \frac{x^4}{4} \right]_0^a \\ &= \frac{3}{2a^3} \times \left[\frac{a^4}{2} - \frac{a^4}{4} \right] \\ &= \frac{3}{8} a \end{aligned}$$

$$\begin{aligned} \bar{y} &= \frac{1}{A} \int y \, dA = \frac{3}{2ab} \int_0^b y (x \, dy) \quad \{\text{taking horizontal strip}\} \\ &= \frac{3}{2ab} \int_0^b \frac{b}{a^2} (a^2 - y^2) x \, dy \\ &= \frac{3}{2a^3} \int_0^b (a^2 x - x y^2) dy \\ &= \frac{3}{2ab} \int_0^b y \cdot a \sqrt{1 - \frac{y^2}{b^2}} \, dy \end{aligned}$$

$$\begin{aligned} \text{Let, } y &= b \sin \theta, \quad dy = b \cos \theta \, d\theta \quad \{\text{limit } 0 \rightarrow b \Rightarrow 0 \rightarrow \frac{\pi}{2}\} \\ \bar{y} &= \frac{3}{2b} \int_0^{\pi/2} b \cos^2 \theta \cdot 2 \sin \theta \cdot \cos \theta \cdot b \, d\theta = \frac{3b^2}{2b} \times 2 \times \frac{1}{2} \times \frac{\pi}{2} = \frac{2b}{5} \end{aligned}$$

Also,

$$\begin{aligned} \bar{y} &= \frac{1}{A} \int y \, dA = \frac{3}{2ab} \int_0^a \left(\frac{y}{a} \right) (y \, dx) \quad \{\text{taking vertical strip}\} \\ &= \frac{3}{4ab} \int_0^a \frac{b^2}{a^4} (a^2 - x^2)^2 dx \\ &= \frac{3b^2}{4a^5 b} \int_0^a (a^4 - 2a^2 x^2 + x^4) dx \\ &= \frac{3b}{4a^5} \left[a^4 x - \frac{2a^2 x^3}{3} + \frac{x^5}{5} \right]_0^a \\ &= \frac{2}{5} b \end{aligned}$$

Q Determine centroid of triangle, semicircle, quarter circle & rectangle by method of integration.

Area under the curve

$$A = \int dA = \int_0^b y dx = \int_0^b \frac{h}{b}(b-x) dx$$

$$= \frac{h}{b} \left[bx - \frac{x^2}{2} \right]_0^b$$

$$= \frac{1}{2} \times b \times h \text{ sq. unit.}$$

$$\bar{X} = \frac{\int x dA}{\int dA} = \frac{\int_0^b x \cdot y dx}{A} = \frac{2 \int_0^b x \cdot \frac{h}{b}(b-x) dx}{b \times h} = \frac{2}{b^2} \left[\frac{bx^2}{2} - \frac{x^3}{3} \right]_0^b$$

$$\bar{X} = \frac{1}{3} \times b \text{ units}$$

$$\bar{Y} = \frac{\int y dA}{\int dA} = \frac{\int_0^h \left(\frac{y}{2}\right)(y dx)}{b \times h} = \frac{1}{2b} \int_0^h \frac{y^2}{b^2} (b-x)^2 dy$$

$$= \frac{h^2}{b^2 \times b \times h} \int_0^b \left[b^2 x - 2bx + x^2 \right] dx$$

$$\bar{Y} = \frac{h}{b^3} \left(\frac{b^3}{3} \right) = \frac{1}{3} h \text{ units}$$

$$\begin{aligned} A &= \int dA = \int_0^{\pi/2} y \cdot dx = \int_0^{\pi/2} \sqrt{a^2 - x^2} dx \\ &= a \int_0^{\pi/2} \sqrt{1 - \sin^2 \theta} d\theta \\ &= a \left[\sin \theta \right]_0^{\pi/2} = a \left[\frac{\pi}{2} \right] = \frac{\pi a^2}{2} \end{aligned}$$

$$\bar{X} = \frac{\int x dA}{\int dA} = \frac{2a \int_0^{\pi/2} x \sqrt{a^2 - x^2} \cdot d\theta}{\pi a^2} = \frac{4}{\pi} a$$

$$\bar{Y} = 0 \text{ unit}$$

$$\int x \sqrt{a^2 - x^2} dx$$

$$\begin{aligned} \bar{Y} &= \frac{\int y dA}{A} = \frac{y}{2\pi a^2} \int_0^{\pi/2} y \cdot y \cdot d\theta = \frac{1}{2\pi a^2} \int_0^{\pi/2} (a^2 - x^2) d\theta \\ &= \frac{1}{2\pi a^2} \left(a^2 \theta - \frac{x^3}{3} \right) = \frac{4a}{3\pi} \text{ unit.} \end{aligned}$$

Rectangle

$$A = \int dA = \int_0^b y dx = h \int_0^b dx = hb$$

$$\bar{X} = \frac{\int x dA}{\int dA} = \frac{\int_0^b x \cdot y dx}{hb} = \frac{h}{hb} \int_0^b \frac{x^2}{2} dx = \frac{h}{2} \text{ unit}$$

$$\bar{Y} = \frac{\int y dA}{\int dA} = \frac{\int_0^h y \cdot x dy}{hb} = \frac{b}{hb} \int_0^h \left[\frac{y^2}{2} \right]_0^h = \frac{h}{2} \text{ unit}$$

Quarter-circle

$$A = \int dA = \int_0^a y \cdot dx = \int_0^a \sqrt{a^2 - x^2} dx$$

$$= \int_0^{\pi/2} \left[\frac{x^2}{2} \sqrt{1 - \sin^2 \theta} + \frac{a^2}{2} \sin \theta \right] d\theta$$

$$A = \frac{a^2}{2} \times \frac{\pi}{2} = \frac{\pi a^2}{4} \text{ sq. unit}$$

$$\bar{X} = \frac{\int x dA}{A} = \frac{4}{\pi a^2} \int_0^a x \cdot y dx = \frac{4}{\pi a^2} \int_0^{\pi/2} x \sqrt{a^2 - x^2} d\theta$$

$$\text{Let, } x = a \sin \theta \Rightarrow dx = a \cos \theta d\theta$$

$$0 \rightarrow a \Rightarrow 0 \rightarrow \pi/2$$

$$\bar{X} = \frac{4}{\pi a^2} \int_0^{\pi/2} a \sin \theta \cdot a \cos \theta \cdot a \cos \theta d\theta = \frac{4a}{\pi} \int_0^{\pi/2} \sin \theta \cos^2 \theta d\theta$$

$$\bar{X} = \frac{4a}{3\pi} \text{ unit}$$

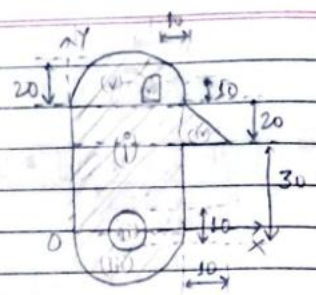
$$\bar{Y} = \frac{\int y dA}{A} = \frac{4}{\pi a^2} \int_0^a y \cdot x dy = \frac{2}{\pi a^2} \int_0^{\pi/2} (a^2 - x^2) d\theta = \frac{2}{\pi a^2} \left[a^2 \theta - \frac{x^3}{3} \right]_0^{\pi/2} = \frac{4a}{3\pi} \text{ unit}$$

$$\bar{Y} = \frac{4a}{3\pi} \text{ unit}$$

Q. Determines the C.G. of given composite figure.

→ Sol:

consider co-ordinate axes



Parts	x_i	y_i	A_i	$x_i A_i$	$y_i A_i$
(1) Rectangle (50x40)	20 (40/2)	25 (50/2)	2000	40000	50000
(2) Circle ($\phi=10$)	20	0	-78.54 (770 ²)	-1570.8	0
(3) Semicircle	20	-8.488 (48/3 π)	628.32	12566.37	-5333.18
(4) Triangle (40+10/3)	43.33 (30+20/3)	36.67 (1/2 x b x h)	100	4333	3667
(5) Semicircle	20	58.48 (50+48/3 π)	628.32	12566.37	36744.15
(6) Quarter-circle	24.24 (20+48/3 π)	54.24 (50+48/3 π)	-78.54	-1903.81	-4260
	ΣA_i		$\Sigma x_i A_i$		$\Sigma y_i A_i$
			= 3199.56	= 65991.13	= 80817.97

Now,

$$\bar{X} = \frac{\Sigma x_i A_i}{\Sigma A_i} = \frac{65991.13}{3199.56} = 20.63 \text{ units}$$

$$\bar{Y} = \frac{\Sigma y_i A_i}{\Sigma A_i} = \frac{80817.97}{3199.56} = 25.26 \text{ units}$$

Moment of Inertia

✓ First moment of area = Area X distance

↑
distance
bw axes &
centroid

✓ If first moment of area is again multiplied by distance, the product is called second moment of area. It is also called second moment of inertia.

$$\text{MOI about } x\text{-axis } (I_{xx}) = \Sigma A_i y_i^2$$

$$I_{xx} = \int y^2 dA$$

$$\text{MOI about } y\text{-axis } (I_{yy}) = \Sigma A_i x_i^2$$

$$I_{yy} = \int x^2 dA$$

Triangle:

$$I_{xx} \text{ (about centroidal } x\text{-axis)} = \frac{bh^3}{36}$$

Passing through centroid

$$I_{yy} \text{ (about centroidal } y\text{-axis)} = \frac{hb^3}{36}$$

Rectangle:

$$I_{xx} = \frac{bh^3}{12}$$

$$I_{yy} = \frac{hb^3}{12}$$

Semicircle:

$$I_{xx} = 0.118r^4$$

$$I_{yy} = \frac{\pi r^4}{8}$$

Quarter-circle:

$$I_{xx} = 0.055r^4$$

$$I_{yy} = 0.055r^4$$

Circle:

$$I_{xx} = \frac{\pi r^4}{4}$$

$$I_{yy} = \frac{\pi r^4}{4}$$

Parallel axis Theorem (Transfer Theorem)

→ State & prove parallel axis theorem.
 If I_{C_y} be the MOI of a plane figure having area 'A' about its centroidal axis, then the moment of inertia about an axis (I_{xx}) which is parallel to centroidal axis and at a distance of 'h' can be obtained by

$$I_{xx} = I_{C_y} + Ah^2$$

→ let us consider a small element with area 'dA' at distance 'y' from centroidal axis.

$$I_{C_y} = \int y^2 dA$$

$$I_{xx} = \int (y+h)^2 dA$$

$$= \int (y^2 + 2hy + h^2) dA$$

$$= \int y^2 dA + 2h \int y dA + h^2 \int dA \quad \left\{ \int y dA = 0, \text{ since first moment of area about centroidal axis is zero} \right\}$$

$$= \int y^2 dA + h^2 \int dA$$

$$I_{xx} = I_{C_y} + Ah^2$$

$$I_{yy} = I_{x_c x_c} + Ah^2$$

$x_c x_c \rightarrow$ centroidal Y-axis

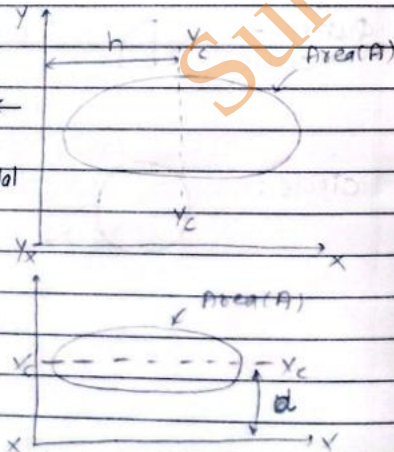
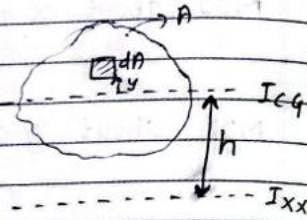
$yy \rightarrow$ axis parallel to centroidal Y-axis

$h \rightarrow$ distance b/w two axis

$$I_{xx} = I_{x_c x_c} + Ah^2$$

$x_c x_c \rightarrow$ Centroidal X-axis

$\int x dA = 0$, First Moment of area about centroidal axis is equal to zero



Perpendicular axis Theorem

→ If I_{xx} and I_{yy} be the moment of inertia of a plane section about two perpendicular axes meeting at O, the moment of inertia I_{zz} about axis z, perpendicular to both previous axes is given by

$$I_{zz} = I_{xx} + I_{yy}$$

Consider a small lamina of area dA at x and y distance from perpendicular axes y and x respectively.

From geometry;

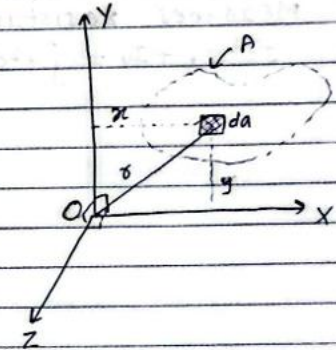
$$r^2 = x^2 + y^2$$

$$I_{zz} = \int r^2 dA$$

$$= \int (x^2 + y^2) dA$$

$$= \int x^2 dA + \int y^2 dA$$

$$\therefore I_{zz} = I_{xx} + I_{yy}$$



Polar axis Theorem

Polar moment of Inertia

→ The pt. of intersection of the axis 'x' and 'y' is called pole, and the axis passing through a point ~~at~~ and perpendicular to the area is called polar axis. The moment of inertia about this axis is polar moment of inertia

$$I_p = I_x + I_y$$

→ Also, denoted by 'J'

→ Measures resistance to twisting.

$$J = I_x + I_y = \int y^2 dA + \int x^2 dA$$

Q. Determine the MOI of the figure as shown.

→ (i) About centroidal axis.

(ii) About given axis.

→ We have,

Centroid of given composite figure,

$$(\bar{x}, \bar{y}) = (116.79, 349.24)$$

Area of figure

Now,

For moment of Inertia about centroid axes:

Component

Parts

1: Triangle ; 2: Rectangle ; 3: Semicircle ; 4: Circle

(i) About Centroidal axis

$$\begin{aligned} I_{x_c x_c} &= (I_{x_c x_c})_1 + (I_{x_c x_c})_2 + (I_{x_c x_c})_3 - (I_{x_c x_c})_4 \\ &= \frac{(100)^3 \times 240}{36} + \frac{(500)^3 \times 240}{12} + 0.11 (120)^4 - \pi \frac{(40)^4}{4} \\ &= 6666666.667 + 25 \times 10^9 + 22809600 - 2010619.29 \\ &= \end{aligned}$$

$$A_1 = \frac{1}{2} \times 240 \times 100 = 12000 \text{ cm}^2$$

$$A_2 = 500 \times 240 = 120000 \text{ cm}^2$$

$$A_3 = \frac{\pi \times (120)^2}{2} = 7200\pi \text{ cm}^2$$

$$A_4 = \pi (40)^2 = 1600\pi \text{ cm}^2$$

$$A = A_1 + A_2 + A_3 - A_4$$

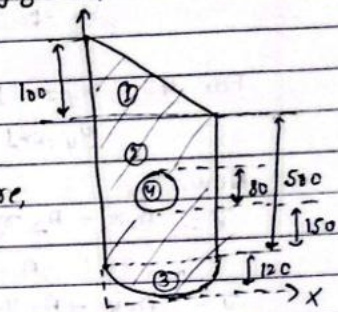
$$A = 149592.92 \text{ cm}^2$$

$$\text{For 1: } y_{\bar{x}_1} = 120 + 500 + \frac{100}{3} = \frac{1960}{3} \text{ cm}$$

$$x_{\bar{y}_1} = \frac{240}{3} = 80 \text{ cm}$$

$$\text{For 2: } x_2 = 240/2 = 120 \text{ cm}$$

$$y_2 = 120 + 500/2 = 370 \text{ cm}$$



For 3: $m_3 = 120\text{cm}$
 $y_3 = \frac{4 \times 120}{3\pi} = \frac{160}{\pi} \text{ cm } 69.07\text{cm}$

For 4: $m_4 = 120\text{cm}$
 $y_4 = 50 + \frac{80}{2} = 90\text{cm}$

Now,

$$\bar{x} = \frac{A_1 x_1 + A_2 x_2 + A_3 x_3 - A_4 x_4}{A_1 + A_2 + A_3 - A_4} = 116.79\text{cm}$$

$$\bar{y} = \frac{A_1 y_1 + A_2 y_2 + A_3 y_3 - A_4 y_4}{A_1 + A_2 + A_3 - A_4} = 349.24\text{cm}$$

Now,

Calculating M.I about centroidal axes.

$$I_{x_c x_c} = \left[\frac{(100)^3 \times 240}{36} + 12000 \left(\frac{160}{3} - 349.24 \right)^2 \right] + \left[\frac{(500)^3 \times 240}{12} + 120000 (370 - 349.24)^2 \right] + \left[0.11 (120)^4 + 7200\pi (69.07 - 349.24)^2 \right] - \left[\frac{\pi (40)^4}{4} + 1600\pi (310 - 349.24)^2 \right]$$

Annotations: "dist. centroidal axis of triangle", "distance of centroid of composite figure from reference axis", "Area of triangle", "Area of rectangle", "dist. of triangle from reference axis", "Figure from reference axis"

$$= 1116339731 + 2551717312 + 1798329848 - 9750385682$$

$$I_{x_c x_c} = 5455636504 \text{ cm}^4$$

$$I_{y_c y_c} = \left[\frac{(100)^3 \times 240^3}{36} + 12000 (80 - 116.79)^2 \right] + \left[\frac{240 \times 500^3}{12} + 120000 (120 - 116.79)^2 \right] + \left[\frac{\pi (120)^4}{8} + 7200\pi (120 - 116.79)^2 \right] - \left[\frac{\pi (40)^4}{4} + 1600\pi (120 - 116.79)^2 \right]$$

$$= 54642049.2 + 250122577.236492 + 81663154.83 - 2062413.354$$

$$= 711479282.7 \text{ cm}^4$$

About reference axes

Using parallel axis theorem,

$$I_{xx} = I_{x_c x_c} + A(H)^2$$

$$= 5455636504 + 149592.92 (349.24)^2$$

$$I_{xx} = 2.37013 \times 10^{10} \text{ cm}^2$$

$$I_{yy} = I_{y_c y_c} + A(H)^2$$

$$= 711479282.7 + 149592.92 (116.79)^2$$

$$= 2753825186 \text{ cm}^2$$

Determine the moment of inertia of inertia of simple plane figure by integration method.

(1) Rectangle

Moment of inertia about

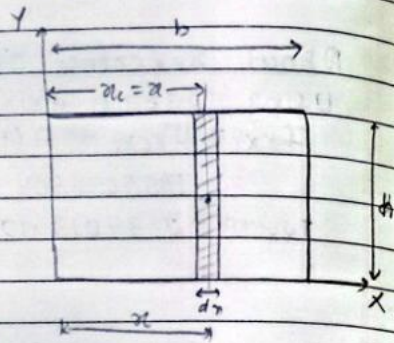
Y-axis (I_{yy})

$$= \int x_c^2 dA$$

$$= \int_0^b x^2 (dh) dx$$

$$= dh \left[\frac{x^3}{3} \right]_0^b$$

$$I_{yy} = h \frac{b^3}{3}$$



(Strip // to Y-axis $\Rightarrow$ M.I. about I_{yy})
(For M.I. about I_{xx} $\Rightarrow$ strip // to X-axis)

For I_{xx}

Moment of inertia about

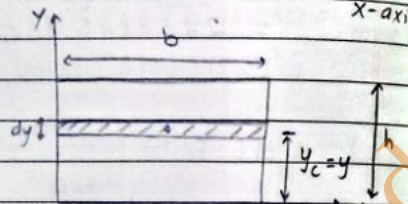
X-axis (I_{xx})

$$= \int y_c^2 dA$$

$$= \int_0^h y^2 (b dy)$$

$$= b \left[\frac{y^3}{3} \right]_0^h$$

$$I_{xx} = \frac{bh^3}{3}$$



About centroidal axis

We have,

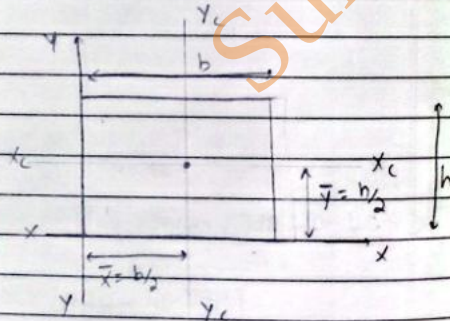
$$I_{xx} = \frac{bh^3}{3}, I_{yy} = \frac{hb^3}{3}$$

$$\text{Area}(A) = bh$$

$$\bar{x} = b/2, \bar{y} = h/2$$

From parallel axis theorem,

$$I_{xx} = I_{x_c x_c} + Ah^2$$



$$\text{or, } \frac{bh^3}{3} = I_{x_c x_c} + A \left(\frac{h}{2} \right)^2$$

$$\text{or, } I_{x_c x_c} = \frac{bh^3}{3} - \frac{bh^3}{4}$$

$$\therefore I_{x_c x_c} = \frac{bh^3}{12}$$

$\{ h \rightarrow \text{distance blw centroidal axis and axis parallel to centroidal axis} = \bar{y} = h/2 \}$

Similarly,

$$I_{yy} = I_{y_c y_c} + Ah^2$$

$$\text{or, } \frac{hb^3}{3} = I_{y_c y_c} + (bh) \left(\frac{b}{2} \right)^2$$

$$\therefore I_{y_c y_c} = \frac{hb^3}{12}$$

M.I. about centroidal Y-axis.

(2) Triangle

Moment of inertia about reference Y-axis

$$(I_{yy}) = \int x_c^2 dA$$

Area of vertical strip

$$= \int_0^b x^2 (y_1 - 0) dx$$

$$= \int_0^b x^2 \frac{h(b-x)}{b} dx$$

$$= \frac{h}{b} \int_0^b (x^2 b - x^3) dx$$

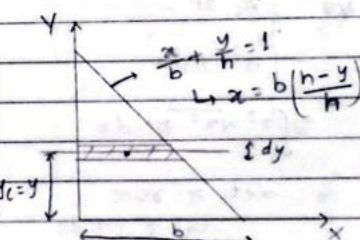
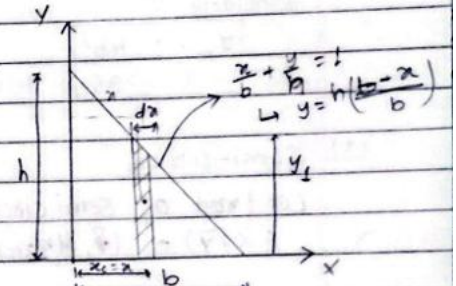
$$= \frac{h}{b} \left[\frac{b x^3}{3} - \frac{x^4}{4} \right]_0^b$$

$$I_{yy} = \frac{1}{12} hb^3$$

Similarly,

$$I_{xx} = \int y_c^2 dA = \int y^2 \frac{b(h-y)}{h} dy$$

$$I_{xx} = \frac{1}{12} bh^3$$



About centroidal axis

Centroid of triangle

$$(\bar{x}, \bar{y}) = \left(\frac{b}{3}, \frac{h}{3}\right)$$

Now,

From parallel axis theorem,

$$I_{xx} = I_{xcxc} + A H^2$$

$$\text{or, } \frac{bh^3}{12} = I_{xcxc} + \left(\frac{1}{2}bh\right)\left(\frac{h}{3}\right)^2$$

$$\text{or, } I_{xcxc} = \frac{bh^3}{12} - \frac{bh^3}{18}$$

$$\therefore I_{xcxc} = \frac{bh^3}{36} \rightarrow \text{M.S. of a triangle (rt-angled) about centroidal x-axis}$$

Similarly,

$$I_{ycyc} = \frac{hb^3}{36}$$

(3) Semi-circle

Centroid of semi circle

$$(\bar{x}, \bar{y}) = \left(\frac{\pi}{2}, \frac{4r}{3\pi}\right)$$

$$\begin{aligned} I_{xcyc} &= \int x_c^2 dA \\ &= \int x^2 y_1 dx \\ &= 2 \int_0^r x^2 \sqrt{r^2 - x^2} dx \end{aligned}$$

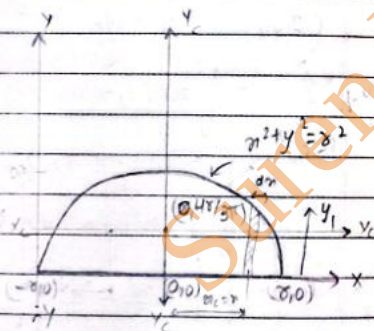
$$\text{Let, } x = r \sin \theta$$

$$dx = r \cos \theta d\theta$$

$$x=0, \theta=0 \mid x=r, \theta=\pi/2$$

$$= 2 \int_0^{\pi/2} r^2 \sin^2 \theta \cdot r \cos \theta \cdot r \cos \theta d\theta$$

$$= 2r^4 \left[\frac{\sin^3 \theta}{3} \right]_0^{\pi/2} = \frac{2r^4 \times \frac{1}{3} \times \frac{\pi}{2}}{2 \times 2!} = \frac{\pi r^4}{8}$$



$$I_{xx} = \int y_c^2 dA$$

$$= 2 \int_0^r y^2(x) dy$$

$$= 2 \int_0^r y^2 \sqrt{r^2 - y^2} dy$$

$$\text{Let, } y = r \sin \theta \Rightarrow dy = r \cos \theta d\theta$$

$$y: 0 \rightarrow r \Rightarrow \theta: 0 \rightarrow \pi/2$$

$$= 2 \int_0^{\pi/2} r^2 \sin^2 \theta \cdot r \cos \theta \cdot r \cos \theta d\theta$$

$$= 2r^4 \left[\frac{\sin^3 \theta}{3} \right]_0^{\pi/2}$$

$$= \frac{\pi r^4}{8}$$

Now, From parallel axis theorem.

$$I_{yy} = I_{ycyc} + A(H)^2 \rightarrow H \rightarrow \text{distance b/w } y_c y_c \text{ and centroidal y-axis, and reference y-axis}$$

$$I_{yy} = \frac{\pi r^4}{8} + \left(\frac{\pi r^2}{2}\right)(r^2)$$

$$I_{yy} = \frac{\pi r^4}{8} + \frac{\pi r^4}{2}$$

$$I_{yy} = \frac{5\pi r^4}{8}$$

Also,

$$I_{xx} = I_{xcxc} + A(H)^2$$

$$\text{or, } \frac{\pi r^4}{8} = I_{xcxc} + \left(\frac{\pi r^2}{2}\right)\left(\frac{4r}{3\pi}\right)^2$$

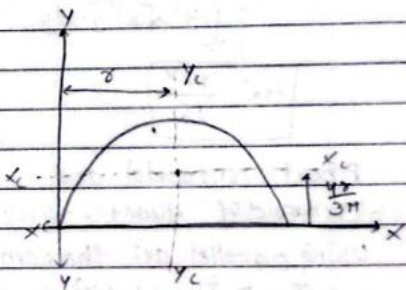
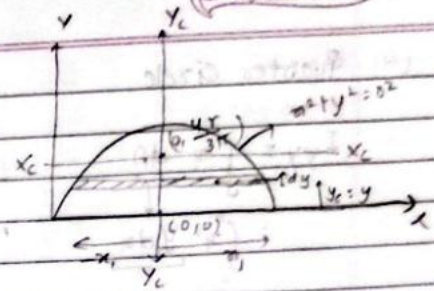
$$\text{or, } I_{xcxc} = \frac{\pi r^4}{8} - \frac{\pi r^2 \times 16r^2}{2 \times 9\pi^2}$$

$$\text{or, } I_{xcxc} = 0.11r^4$$

$\therefore$ For semi circle, About centroidal axis: $I_{xcxc} = 0.11r^4$

$$I_{ycyc} = \frac{\pi r^4}{8}$$

About reference axis: $I_{xx} = \frac{\pi r^4}{8}$, $I_{yy} = \frac{5\pi r^4}{8}$

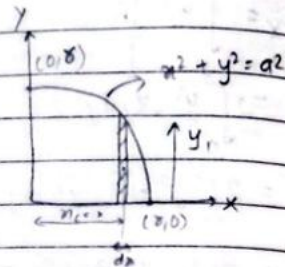


(4) Quarter circle

$$I_{yy} = \int x^2 dA$$

$$= \int_0^{\pi/2} r^2 y dr$$

$$= \int_0^{\pi/2} r^2 \sqrt{a^2 - r^2} dr$$



Let, $r = a \sin \theta \Rightarrow dr = a \cos \theta d\theta$

$r: 0 \rightarrow a \quad \theta: 0 \rightarrow \pi/2$

$$= \int_0^{\pi/2} a^2 \sin^2 \theta \cdot a \cos \theta \cdot a \cos \theta d\theta$$

$$= a^4 \frac{\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4} \right]_0^{\pi/2}}{2 \cdot \frac{a}{2}} = \frac{a^4 \times \frac{1}{2} \times \frac{1}{2} \times \pi}{2 \times 2} = \frac{\pi a^4}{16}$$

$$\boxed{I_{yy} = \frac{\pi a^4}{16}}$$

Similarly,

$$I_{xx} = \int y_c^2 dA$$

$$= \int_0^a y^2 r dr$$

$$= \int_0^a y^2 \sqrt{a^2 - y^2} dy$$

$$\boxed{I_{xx} = \frac{\pi a^4}{16}}$$

About centroidal axis

Centroid of quarter-circle $(\bar{x}, \bar{y}) = \left(\frac{4r}{3\pi}, \frac{4r}{3\pi} \right)$

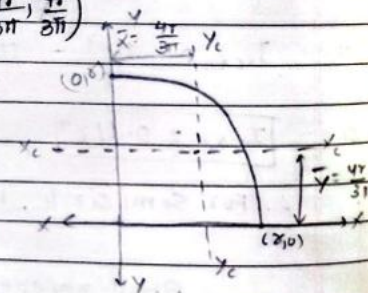
Using parallel axis theorem,

$$I_{yy} = I_{y_c y_c} + A H^2$$

$$\text{or } \frac{\pi a^4}{16} = I_{y_c y_c} + \left(\frac{\pi a^2}{4} \right) \left(\frac{4a}{3\pi} \right)^2$$

$$\text{or } I_{y_c y_c} = \frac{\pi a^4}{16} - \frac{\pi a^2}{4} \times \frac{16a^2}{9\pi^2}$$

$$\boxed{I_{y_c y_c} = 0.055 a^4}$$



Similarly,

$$I_{xx} = I_{x_c x_c} + A H^2$$

$$\text{or } \frac{\pi a^4}{16} = I_{x_c x_c} + \left(\frac{\pi a^2}{4} \right) \left(\frac{4a}{3\pi} \right)^2$$

$$\therefore \boxed{I_{x_c x_c} = 0.055 a^4}$$

† Moment of inertia by Integration

Q

We have,

$$Area = \int dA$$

$$= \int_0^a y dr$$

$$= \int_0^a \frac{b}{a^2} (a^2 - r^2) dr$$

$$A = \frac{2}{3} ab$$

$$\text{Centroid: } (\bar{x}, \bar{y}) = \left(\frac{3}{8} a, \frac{2b}{5} \right)$$

Now,

Moment of Inertia About reference y-axis

$$\text{Fig (i)} \quad (I_{yy}) = \int r_c^2 dA = \int_0^a r^2 (y_1 dr)$$

$$I_{yy} = \int_0^a \frac{r^2 b}{a^2} (a^2 - r^2) dr = \frac{b}{a^2} \int_0^a [a^2 r^2 - r^4] dr$$

$$I_{yy} = \frac{b}{a^2} \left[\frac{a^2 r^3}{3} - \frac{r^5}{5} \right]_0^a$$

$$\therefore \boxed{I_{yy} = \frac{2}{15} a^3 b}$$

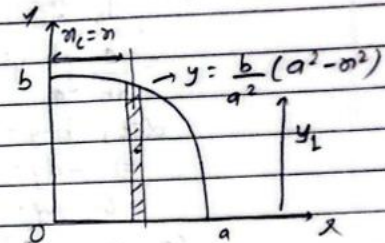


Fig (i)

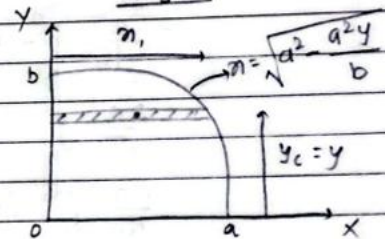


Fig (ii)

Fig(ii).

Moment of inertia about reference x-axis

$$I_{xx} = \int y^2 dA$$

$$= \int_0^b y^2 x_1 dy$$

$$= \int_0^b y^2 \sqrt{a^2 - \frac{a^2 y}{b}} dy$$

$$= \int_0^b y^2 \sqrt{a^2 b - a^2 y} dy$$

$$= \frac{a}{\sqrt{b}} \int_0^b y^2 \sqrt{b-y} dy$$

$$\text{let, } b-y = z^2$$

$$\Rightarrow -dy = 2z dz$$

$$\text{or, } dy = -2z dz$$

$$\text{So, For: } y: 0 \rightarrow b \quad z: \sqrt{b} \rightarrow 0$$

$$= \frac{a}{\sqrt{b}} \int_{\sqrt{b}}^0 -(b-z^2)^2 z 2z dz$$

$$= \frac{2a}{\sqrt{b}} \int_0^{\sqrt{b}} (b^2 - 2bz^2 + z^4) z^2 dz$$

$$= \frac{2a}{\sqrt{b}} \int_0^{\sqrt{b}} (bz^2 - 2bz^4 + z^6) dz$$

$$= \frac{2a}{\sqrt{b}} \left[\frac{bz^3}{3} - \frac{2bz^5}{5} + \frac{z^7}{7} \right]_0^{\sqrt{b}}$$

$$= \frac{2a}{\sqrt{b}} \left[\frac{b^3 \sqrt{b}}{3} - \frac{2b^3 \sqrt{b}}{5} + \frac{b^3 \sqrt{b}}{7} \right]$$

$$= \frac{16}{105} ab^3$$

$$\therefore I_{xx} = \frac{16}{105} ab^3$$

From parallel axis theorem,

$$I_{xx} = I_{xcxc} + A(H)^2$$

$$\text{or, } \frac{16}{105} ab^3 = I_{xcxc} + \frac{2}{3} ab \left(\frac{2}{3} b \right)^2$$

$$\text{or, } \frac{16}{105} ab^3 = I_{xcxc} + \frac{2}{3} ab \times \frac{4}{25} b^2$$

$$\text{or, } I_{xcxc} = \frac{16}{105} ab^3 - \frac{8}{75} ab^3$$

$$\therefore I_{xcxc} = \frac{8}{175} ab^3 \rightarrow \text{About centroidal x-axis}$$

Also,

$$I_{yy} = I_{ycyc} + A(H)^2$$

$$\text{or, } \frac{2}{15} a^3 b = I_{ycyc} + \frac{2}{3} ab \left(\frac{2}{3} a \right)^2$$

$$\text{or, } I_{ycyc} = \frac{2}{15} a^3 b - \frac{2}{3} \times \frac{4}{9} a^3 b$$

$$\therefore I_{ycyc} = \frac{19}{480} a^3 b \rightarrow \text{About centroidal y-axis}$$

$$I_{xx} = \int \frac{y^3}{3} dx$$

$$= \frac{1}{3} \int_0^a \left[\frac{b}{a^2} (a^2 - x^2) \right]^3 dx$$

$$= \frac{b^3}{3a^6} \int_0^a [a^6 - x^6 - 3a^4 x^2 + 3a^2 x^4] dx$$

$$= \frac{b^3}{396} \left[a^6 x - \frac{x^7}{7} - \frac{3a^4 x^3}{3} + \frac{3a^2 x^5}{5} \right]_0^a$$

$$= I_{xx} = \frac{16}{105} b^3 a$$

Q Determine the moment of inertia of given composite figure.

→ Solution

Let.

Rectangle : (i)

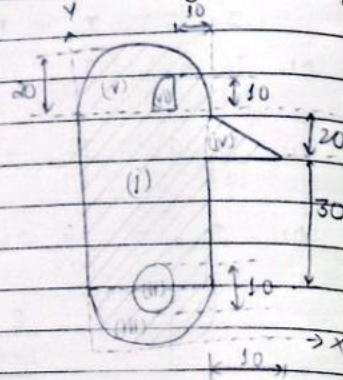
Circle : (ii)

Semicircle : (iii)

Triangle : (iv)

Semi-circle : (v)

Quarter-circle : (vi)



Component parts	x_i	y_i	A
(i)	$40/2 = 20$	$20 + 50/2 = 45$	$40 \times 50 = 2000$
(ii)	$40/2 = 20$	20	$\pi(5)^2 = 25\pi$
(iii)	$40/2 = 20$	$20 - \frac{4 \times 20}{3\pi} = 11.52$	$\frac{\pi(20)^2}{2} = 200\pi$
(iv)	$40 + \frac{10}{3} = \frac{130}{3}$	$20 + 30 + \frac{20}{3} = \frac{170}{3}$	$\frac{1}{2} \times 10 \times 20 = 100$
(v)	$40/2 = 20$	$20 + 50 + \frac{4 \times 20}{3\pi} = 78.92$	$\frac{\pi(20)^2}{2} = 200\pi$
(vi)	$40 - 10 - \frac{4 \times 10}{3\pi} = 25.76$	$20 + 50 + \frac{4 \times 10}{3\pi} = 74.24$	$\frac{\pi(10)^2}{4} = 25\pi$

Now,

$$\bar{X} = x_1 A_1 + x_2 A_2 + x_3 A_3 + x_4 A_4 + x_5 A_5 - x_6 A_6$$

$$A_1 - A_2 + A_3 + A_4 + A_5 - A_6$$

$$= \frac{65872.093}{3199.56}$$

$$= 20.588 \text{ cm}$$

$$\bar{Y} = y_1 A_1 - y_2 A_2 + y_3 A_3 + y_4 A_4 + y_5 A_5 - y_6 A_6$$

$$A_1 - A_2 + A_3 + A_4 + A_5 - A_6$$

$$= \frac{145090.2023}{3199.56}$$

$$= 45.35 \text{ cm}$$

Circle: $I_c = \frac{\pi D^4}{64} = \frac{\pi (20)^4}{64} = \frac{\pi 8^4}{8}$

Now,

Calculating M.I about centroidal axes:

Centroidal X-axis

$$I_{x_c} = \left[\frac{40 \times 50^3}{12} + 2000(20.588 - 45)^2 \right]$$

$$- \left[\frac{\pi(5)^4}{8} + 25\pi(45.35 - 20)^2 \right]$$

$$+ \left[\frac{\pi(20)^4}{8} + 200\pi(45.35 - 11.52)^2 \right]$$

$$+ \left[\frac{10 \times 20^3}{36} + 100(45.35 - 56.67)^2 \right]$$

$$+ \left[\frac{\pi(20)^4}{8} + 200\pi(45.35 - 78.92)^2 \right]$$

$$- \left[\frac{\pi(10)^4}{8} + 25\pi(45.35 - 74.24)^2 \right]$$

$$= 416911.67 - 50962.33 + 781922.87 + 15036.46$$

$$+ 77091.22 - 67515.35 + 66101.85$$

$$= 1866305.54 \text{ cm}^4$$

Centroidal Y-axis

$$I_{y_c} = \left[\frac{h b^3}{12} + A_1(x_1 - \bar{x})^2 \right] - \left[\frac{\pi r^4}{4} + A_2(\bar{x}_2 - \bar{x})^2 \right]$$

$$+ \left[\frac{\pi r^4}{8} + A_3(\bar{x}_3 - \bar{x})^2 \right] + \left[\frac{h b^3}{36} + A_4(\bar{x}_4 - \bar{x})^2 \right]$$

$$+ \left[\frac{\pi r^4}{8} + A_5(\bar{x}_5 - \bar{x})^2 \right] - \left[\frac{\pi r^4}{8} + A_6(\bar{x}_6 - \bar{x})^2 \right]$$

$$= \left[\frac{50 \times 40^3}{12} + 2000(20 - 20.588)^2 \right] - \left[\frac{\pi(5)^4}{4} + 25\pi(20 - 20.588)^2 \right]$$

$$+ \left[\frac{\pi(20)^4}{8} + 200\pi(20 - 20.588)^2 \right] + \left[\frac{10 \times 20^3}{36} + 100(43.33 - 20.588)^2 \right]$$

$$+ \left[\frac{\pi(20)^4}{8} + 200\pi(20 - 20.588)^2 \right] - \left[\frac{\pi(10)^4}{8} + 25\pi(25.76 - 20.588)^2 \right]$$

$$= 267358.15 - 518.03 + 63049.09 + 53942.08 + 63049.09$$

$$- 4064.40 \quad 2650.91$$

$$= 442815.98 \text{ cm}^4 \quad 444229.47 \text{ cm}^4$$

Now,

About reference axis,

$$I_{xx} = I_{x_c x_c} + A h^2$$

$$I_{xx} = 1866305.54 + 3199.56 \times (45.35)^2$$

$$1777255.334$$

$$I_{xx} = 8446592.63 \text{ cm}^4 \quad 8357542.42 \text{ cm}^4$$

$$I_{yy} = I_{y_c y_c} + A h^2$$

$$= 442815.98 + 3199.56 \times (20.588)^2$$

$$I_{yy} = 9802776.506 \text{ cm}^4 \quad 1800413.35 \text{ cm}^4$$

Radius of gyration:

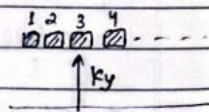
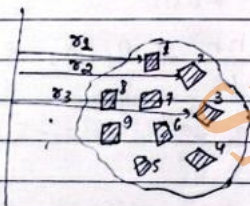
The radius of gyration of any lamina about a given axis may be defined as a distance from the given axis at which all the elementary parts of the lamina would have to be placed so as not to alter the moment of Inertia about the given axis.

Mathematically,

$$I_{xx} = k_{xx}^2 A$$

$$\therefore k_{xx} = \sqrt{\frac{I_{xx}}{A}}$$

$$\& k_{yy} = \sqrt{\frac{I_{yy}}{A}}$$



* Physical meaning of 'I'.

Q. Determine moment of inertia

$$I_{xx} = \int y^2 dA$$

$$= \int y^2 \cdot (2x) \cdot dy$$

$$= \int_0^2 x^2 (2x - x^2) dx$$

$$= \int_0^2 (2x^3 - x^4) dx$$

$$= \left[\frac{2x^4}{4} - \frac{x^5}{5} \right]_0^2$$

$$= \frac{32}{4} - \frac{32}{5}$$

$$= \frac{8}{5} \text{ m}^4$$

$$I_{xx} = \int y_c^2 dA = \int \left(\frac{y}{2}\right)^2 y dx$$

$$= \int y^2 dx = \int \frac{y^3}{4} dx$$

$$= \int_0^2 \frac{(2x - x^2)^3}{4} dx$$

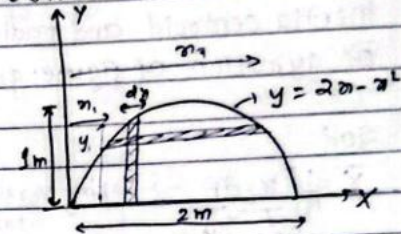
$$= \frac{1}{4} \int_0^2 (8x^3 - 12x^4 + 6x^5) dx$$

$$= \frac{1}{4} \left[\frac{8x^4}{4} - \frac{12x^5}{5} + \frac{6x^6}{6} \right]_0^2$$

$$= \frac{1}{4} \left[\frac{8x^4}{4} - \frac{12x^5}{5} + \frac{6x^6}{6} \right]_0^2$$

$$= \frac{1}{4} \left[\frac{8x^4}{4} - \frac{12x^5}{5} + \frac{6x^6}{6} \right]_0^2$$

$$= \frac{8}{35} \text{ m}^4$$



Q. Determine moment of inertia, centroid and radius of gyration of figure given

→ Soln:

$\bar{X} = \frac{1}{A} \int x_c dA$ { Taking vertical strip }

$$= 6 \int_0^1 \frac{x}{2} (y_2 - y_1) dx$$

$$= 6 \int_0^1 \frac{x}{2} (x - x^2) dx$$

$$= 6 \int_0^1 \frac{1}{2} (x^2 - x^3) dx$$

$$= 6 \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1$$

$$= \frac{6}{12} \text{ units} = \frac{1}{2} \text{ units}$$

$\bar{Y} = \frac{1}{A} \int y_c dA$ { Taking vertical strip } OR

$$= 6 \int_0^1 \left(y_1 + \frac{y_2 - y_1}{2} \right) (y_2 - y_1) dx$$

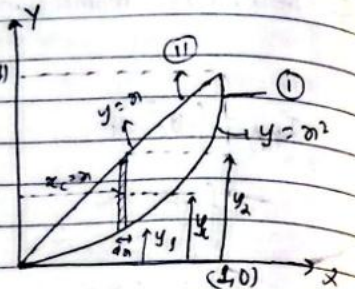
$$= 6 \int_0^1 \frac{(y_1 + y_2)(y_2 - y_1)}{2} dx$$

$$= 6 \int_0^1 \frac{(y_2^2 - y_1^2)}{2} dx$$

$$= 6 \int_0^1 \frac{(x^2 - x^4)}{2} dx$$

$$= \frac{6 \times 1}{2} \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1$$

$$= \frac{6}{15} \text{ units} = \frac{2}{5} \text{ units}$$

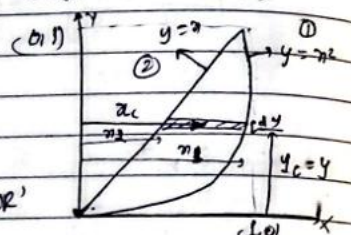


$$\text{eqn (i)}: y = x^2 \rightarrow y_1$$

$$\text{eqn (ii)}: y = x \rightarrow y_2$$

Solving

$$(x, y) = (0, 0) \text{ \& \; } (1, 1)$$



$\bar{Y} = \frac{1}{A} \int y_c dA$ { Taking horizontal strip }

$$= 6 \int_0^1 y (x_2 + x_1) dy$$

$$= 6 \int_0^1 y (y + \sqrt{y}) dy$$

$$= 6 \int_0^1 (y^2 + y^{3/2}) dy$$

$$= 6 \left[\frac{y^3}{3} + \frac{2y^{5/2}}{5} \right]_0^1$$

$$= \frac{6}{15} \text{ units}$$

$$= \frac{2}{5} \text{ units}$$

$$A = \int dA = \int_0^1 (y_2 - y_1) dx = \int_0^1 (x - x^2) dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6} \text{ unit}^2$$

$$\therefore \text{Centroid } (\bar{X}, \bar{Y}) = \left(\frac{1}{2}, \frac{2}{5} \right)$$

Now,

Moment of inertia

M.I. about reference Y axis:

$I_{yy} = \int x_c^2 dA$ { Taking vertical strip fig (i) }

$$= \int_0^1 x^2 (y_2 - y_1) dx$$

$$= \int_0^1 x^2 (x - x^2) dx$$

$$= \int_0^1 (x^3 - x^4) dx$$

$$= \left[\frac{x^4}{4} - \frac{x^5}{5} \right]_0^1$$

$$= \frac{1}{20} \text{ units}^4 \quad \therefore \boxed{I_{yy} = \frac{1}{20} \text{ units}^4}$$

OR

$I_{yy} = \int x_c^2 dA$ { Taking horizontal strip fig (ii) }

$$= \int_0^1 \left(\frac{x_1 + x_2}{2} \right)^2 (x_2 - x_1) dy$$

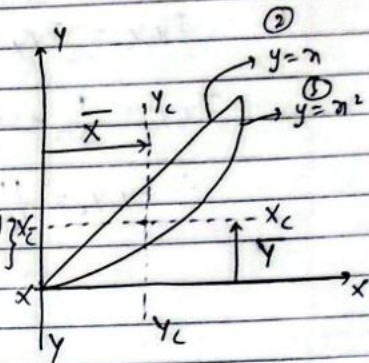
$$= \frac{1}{2} \int_0^1 (x_1^2 - x_2^2) dy = \frac{1}{4} \int_0^1 (x_1^2 - x_2^2) (x_1 + x_2) dy$$

$$= \frac{1}{2} \int_0^1 (y - y^2) dy = \frac{1}{4} \int_0^1 y^3 (y - y^2) (\sqrt{y} + y) dy$$

$$= \frac{1}{2} \left[\frac{y^2}{2} - \frac{y^3}{3} \right]_0^1 = \frac{1}{4} \int_0^1 (y^{3/2} + y^2 - y^{5/2} - y^3) dy$$

$$= \frac{1}{4} \left[\frac{2y^{5/2}}{5} + \frac{y^3}{3} - \frac{2y^{7/2}}{7} - \frac{y^4}{4} \right]_0^1$$

$$= \frac{83}{1680} \approx \frac{1}{20} \text{ units}^4$$



Using parallel axis theorem,

M.I. about centroidal y axis:

$$I_{y_c y_c} = I_{yy} + A h^2 \quad h \rightarrow \text{distance of centroidal y-axis from reference y-axis}$$

$$I_{y_c y_c} = \frac{1}{20} + \frac{1}{6} \times \left(\frac{1}{2}\right)^2$$

$$\therefore I_{y_c y_c} = \frac{11}{120} \text{ units}^4$$

M.I. about reference x-axis:

$$I_{xx} = \int y_c^2 dA \quad \{\text{taking horizontal strip}\}$$

$$= \int_0^1 y^2 (x_1 - x_2) dy$$

$$= \int_0^1 y^2 (\sqrt{y} - y) dy$$

$$= \int_0^1 (y^{5/2} - y^3) dy$$

$$= \left[\frac{2y^{7/2}}{7} - \frac{y^4}{4} \right]_0^1$$

$$= \frac{1}{28} \text{ units}^4$$

OR $I_{xx} = \int y_c^2 dA \quad \{\text{taking vertical strip}\}$

$$= \int_0^1 \left(\frac{y_1 + y_2}{2}\right)^2 (y_2 - y_1) dx$$

$$= \frac{1}{4} \int_0^1 (y_1^2 - y_2^2) (y_1 + y_2) dx$$

$$= \frac{1}{4} \int_0^1 (x^2 - x^4) (x + x^2) dx$$

$$= \frac{1}{4} \int_0^1 (x^3 + x^4 - x^5 - x^6) dx$$

$$= \frac{1}{4} \left[\frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} - \frac{x^7}{7} \right]_0^1$$

$$= \frac{59}{1680}$$

$$\approx \frac{1}{28} \text{ units}^4$$

Using parallel axis theorem

M.I. about centroidal x-axis:

$$I_{x_c x_c} = I_{xx} + A h^2 \quad h \rightarrow \text{distance of } x_c \text{ from } x\text{-axis}$$

$$I_{x_c x_c} = \frac{1}{28} + \frac{1}{6} \left(\frac{2}{5}\right)^2$$

$$\therefore I_{x_c x_c} = \frac{131}{2100} \text{ units}^4$$

Now,

Radius of gyration:

$$K_{xx} = \sqrt{\frac{I_{xx}}{A}} = \sqrt{\frac{1/28}{1/6}} = \frac{\sqrt{42}}{34} = 0.463 \text{ units}$$

$$K_{yy} = \sqrt{\frac{I_{yy}}{A}} = \sqrt{\frac{1/20}{1/6}} = \frac{\sqrt{30}}{10} = 0.548 \text{ units}$$

Kinematics → Branch of Dynamics which deals about relationship between position, velocity & acceleration with time (i.e. position, velocity & accⁿ are function of time)
 $x=f(t)$, $v=f(t)$, $a=f(t)$.

Kinetic:- Branch of dynamics which deals about relationship of force and acceleration (i.e. Newton's 2nd law is mostly used under this, $F=ma$).

DYNAMICS

- Dynamics → $f(t)$ {Parameters: $f(t)$ }
 → Two types:
 (i) Kinematics: without ref. to cause
 (ii) kinetics: study with ref. to cause (force)

Kinematics of Particle

↳ (i) Rectilinear motion of particle

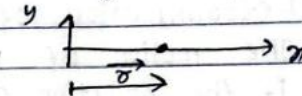
↳ (ii) Curvilinear motion of particle

(i) Rectilinear motion of Particle

A particle moving in straight line is said to be in rectilinear motion

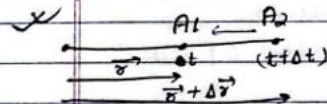
• Position, velocity and acceleration

Consider a particle moving in a st. line in x -axis, then the position of particle is given by

$$\vec{r} = x\hat{i} \quad \text{--- (1)}$$


At time t , particle is at position A_1 and at time $(t+\Delta t)$, it reaches to new position A_2 and position vector as $\vec{r}$ and $(\vec{r}+\Delta\vec{r})$.

$$\begin{aligned} \text{Average acceleration } (\vec{v}_{av}) &= \frac{(\vec{r}+\Delta\vec{r}) - \vec{r}}{(t+\Delta t) - t} \\ &= \frac{\Delta\vec{r}}{\Delta t} \end{aligned}$$



If $\Delta t \rightarrow 0$, the position A_2 approaches A_1 and this is instantaneous velocity of particle at A_1 .

$$\therefore \vec{v}_{\text{instantaneous}} = \lim_{\Delta t \rightarrow 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} = \frac{dx}{dt} \hat{i}$$

Similarly, $t \rightarrow \vec{v}$ & $t + \Delta t \rightarrow \vec{v} + \Delta \vec{v}$
 Average acceleration = $\frac{\Delta \vec{v}}{\Delta t}$

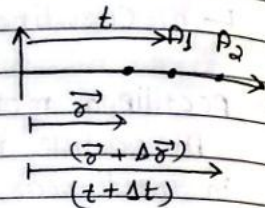
Instantaneous acceleration,
 $\vec{a} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{v}}{\Delta t}$
 $= \frac{d\vec{v}}{dt} = \frac{d^2 \vec{r}}{dt^2} = \frac{d^2 x}{dt^2} \vec{i}$

Scalarly,

$$v = \frac{dx}{dt}, \quad a = \frac{dv}{dt}$$

$$a = \frac{dv}{dx} \times \frac{dx}{dt}$$

$$\therefore \boxed{a = v \frac{dv}{dx}}$$



Determination of motion of particle.

The motion of particle is said to be known if the position of particle for every value of time t is known. However, motion is seldom defined by relation between position (x) and time (t). More often, the condition of motion are specified by the type of acceleration that the particle possess.

• Three common classes of motion:-

(1) $a = f(t)$, acceleration is function of time

$$a = f(t) = \frac{dv}{dt}$$

$$\therefore dx = a dt = f(t) dt$$

Integrating

$$\int_{v_0}^v dv = \int f(t) dt$$

$$\text{or, } v - v_0 = \int_0^t f(t) dt$$

$$\boxed{v = v_0 + \int_0^t f(t) dt}$$

✓ velocity is rate of change of position

$$v = \frac{dx}{dt}$$

$$dx = v dt$$

Integrating

$$\int_{x_0}^x dx = \int_0^t v dt$$

$$\text{or, } x = x_0 + \int_0^t v dt$$

$$\therefore x = x_0 + \int_0^t \left[v_0 + \int_0^t f(t) dt \right] dt$$

Special cases

- $a = 0$, accelⁿ of particle is zero for every value of t (Constant velocity)
- $a = \text{constant}$, acceleration of particle is constant for every value of t

(a) $a = 0$, This is also known as uniform rectilinear motion

When $a = 0$, $v = \text{constant}$

$$\frac{dx}{dt} = v = \text{constant}$$

$$dx = v dt$$

Integrating

$$\int_{x_0}^x dx = \int_{t_0}^t v dt$$

or, $x - x_0 = v(t - t_0)$

or, $x = x_0 + v(t - t_0) \rightarrow$ equation of Motion

(b) $a = \text{constant}$.

This is also known as uniformly accelerated rectilinear motion.

$a = \text{constant}$.

$$\frac{dv}{dt} = a = \text{constant}$$

$$dv = a dt$$

Integrating,

$$\int_{v_0}^v dv = \int_0^t a dt$$

$$v - v_0 = at$$

$$\boxed{v = v_0 + at} \quad (1)$$

Also,

$$v = \frac{dx}{dt} \Rightarrow v_0 + at$$

Integrating,

$$\text{or } x - x_0 = v_0 t + \frac{at^2}{2}$$

$$\boxed{x = x_0 + v_0 t + \frac{at^2}{2}} \quad (2)$$

Again,

$$v \frac{dv}{dx} = a = \text{constant gives}$$

$$\text{or } \int_{v_0}^v v dv = \int_{x_0}^x a dx$$

$$\int_{x_0}^x dx = \int_0^t (v_0 + at) dt$$

$$x - x_0 = v_0 t + \frac{at^2}{2}$$

$$x = x_0 + v_0 t + \frac{at^2}{2}$$

$$\text{or } \frac{v^2 - v_0^2}{2} = a(x - x_0)$$

$$\text{or } \boxed{v^2 = v_0^2 + 2a(x - x_0)} \quad (3)$$

(2) $a = f(x)$.

$$a = v \frac{dv}{dx}$$

$$\text{or } v dv = f(x) dx$$

Integrating,

$$\text{or } \frac{v^2 - v_0^2}{2} = \int_{x_0}^x f(x) dx$$

$$\text{or } \boxed{v^2 = v_0^2 + 2 \int_{x_0}^x f(x) dx}$$

Again,

$$v = \frac{dx}{dt} \Rightarrow dx = v dt$$

Integrating,

$$\text{or } x - x_0 = \int_0^t v dt$$

$$\text{or } x = x_0 + \int_0^t [v_0^2 + 2 \int_{x_0}^x f(x) dx]^{1/2} dt$$

(3) $a = f(v)$

$$\text{or } a = v \frac{dv}{dx}$$

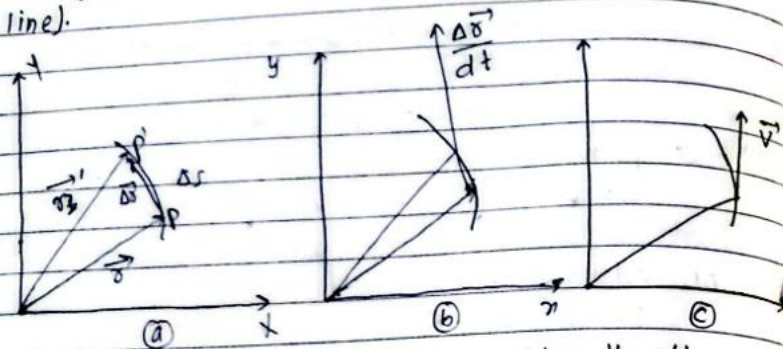
$$\text{or } dx = \frac{v dv}{f(v)}$$

Integrating

$$\text{or } \int_{x_0}^x dx = \int_{v_0}^v \frac{v dv}{f(v)}$$

$$\therefore \boxed{x = x_0 + \int_{v_0}^v \frac{v dv}{f(v)}}$$

Position, velocity and acceleration of a particle moving in curvilinear path (other than straight line).



When a particle moves in a curved path other than straight line, it is called curvilinear motion.

To specify the position of particle, XY reference axis is chosen.

At time t , the position of particle is at P , where its position vector is $\vec{r}$. At time $(t + \Delta t)$, a particle occupies new position P' & its position vector be $\vec{r}'$ that means $\Delta \vec{r}$ represents the change in direction & magnitude of particle.

At t , $\vec{r}$
 $t + \Delta t$, $\vec{r}'$

Avg. velocity, $\vec{v}_{avg} = \frac{\Delta \vec{r}}{\Delta t}$

$\vec{v}_{instantaneous} = \frac{d\vec{r}}{dt}$

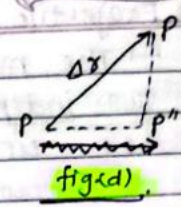
As $\Delta \vec{r}$ and Δt becomes shorter, P and P' get closer and $\vec{v}$ is tangent to the path of a particle in curvilinear motion.

$$\vec{v} = \lim_{\Delta t \rightarrow 0} \frac{\vec{r}' - \vec{r}}{\Delta t}$$

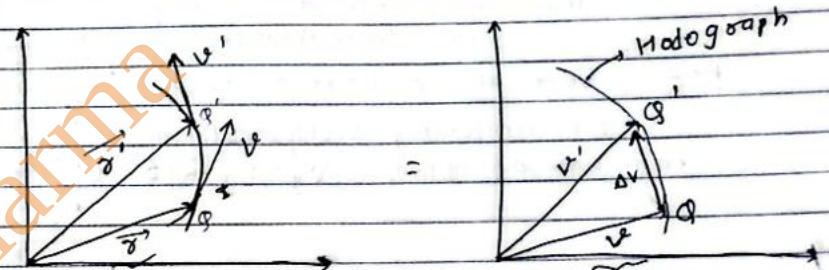
$$= \lim_{\Delta t \rightarrow 0} \left(\frac{\vec{r}' - \vec{r}}{\Delta t} \right)$$

$$= v_x \hat{i} + v_y \hat{j}$$

$$\therefore v = \sqrt{v_x^2 + v_y^2} \text{ (Magnitude)}$$



$$\frac{d\vec{r}}{dt} = v_x \hat{i} + v_y \hat{j}$$



Hodograph $\rightarrow$ not the path of motion
 $\hookrightarrow$ curve drawn by locus of velocities at each pt.

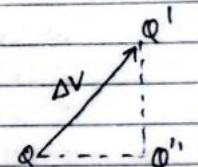
Similarly,

$$a_{avg} = \frac{\Delta \vec{v}}{\Delta t}$$

$$a_{instantaneous} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{v}}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\vec{v}' - \vec{v}}{\Delta t}$$

$$= \lim_{\Delta t \rightarrow 0} \left(\frac{\vec{v}' - \vec{v}}{\Delta t} \right)$$

$$\text{or, } \vec{a} = a_x \hat{i} + a_y \hat{j}$$



fig

Projectile motion

→ The motion of projectile can be replaced by two independent rectilinear motion.
i.e. uniformly accelerated motion (vertical)
& uniform motion ($a=0$) (horizontal)

* Illustrate

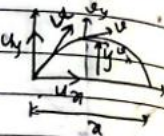
→ For a projectile motion, horizontal acceleration (a_x) = 0
i.e. uniform rectilinear motion

Thus, horizontal velocity of particle remains same throughout the motion $v_x = u_x = \text{const.}$

And, vertical acceleration (a_y) = g (downward)

i.e. uniformly accelerated motion. constant

From eqn of motion, $v_y^2 = u_y^2 + 2g\delta$



Q. The motion of a vibrating particle is defined by the equation

$$x = 100 \sin \pi t \quad \text{positions}$$

$$\& y = 25 \cos 2\pi t$$

where x & y are expressed in mm and t in second

Determine velocity and acceleration when $t=1$ second. Also, Determine the path of the particle

→ Sol:-

$$\dot{x} = v_x = \frac{dx}{dt} = 100\pi \cos \pi t$$

$$\dot{y} = v_y = \frac{dy}{dt} = -50\pi \sin 2\pi t$$

$$\ddot{x} = a_x = \frac{d^2x}{dt^2} = -100\pi^2 \sin \pi t$$

$$\ddot{y} = a_y = \frac{d^2y}{dt^2} = -100\pi^2 \cos 2\pi t$$

path of the particle is
parabola
 $x^2 + 200y = 5000$
 $100\pi^2$

Magnitude

$$\text{At } t=1, |\vec{v}| = \sqrt{v_x^2 + v_y^2} = \sqrt{(100\pi \cos \pi)^2 + (50\pi \sin 2\pi)^2}$$

$$= 100\pi \text{ mm/s}$$

Direction,

$$\theta = \tan^{-1} \left(\frac{v_y}{v_x} \right) = 0^\circ$$

Magnitude,

$$\text{At } t=1, |\vec{a}| = \sqrt{a_x^2 + a_y^2} = \sqrt{(-100\pi^2 \sin \pi)^2 + (100\pi^2 \cos 2\pi)^2}$$

$$= 100\pi^2 \text{ mm/s}^2$$

→ Path of particle

$$y = 25 \cos 2\pi t$$

$$y = 25 \{ 1 - 2 \sin^2 \pi t \}$$

$$y = 25 \left\{ 1 - 2 \cdot \frac{x^2}{100^2} \right\}$$

$$\text{or } y = \frac{25(10000 - x^2)}{10000}$$

$$\text{or } 200y = 25 - \frac{25x^2}{5000}$$

$$\text{or } 200y = 5000 - x^2$$

$$\text{or } x^2 + 200y = 5000 \rightarrow \text{parabolic.}$$

14

$$v_x = 2 \cos t$$

$$v_y = \sin t$$

Initially, it is known that both x & y coordinates are 0. And determine the path of particle & total acceleration at 2 sec.

→ Sol:

$$v_x = 2 \cos t$$

$$v_y = \sin t$$

Now,

$$a_x = \frac{dv_x}{dt} = -2 \sin t$$

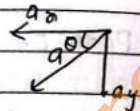
$$a_y = \frac{dv_y}{dt} = \cos t$$

At, $t = 2 \text{ sec}$, $a_x = -1.82 \text{ m/s}^2$

$$a_y = -0.42 \text{ m/s}^2$$

Thus, $|a| = \sqrt{a_x^2 + a_y^2}$
 $= 1.87 \text{ m/s}^2$

$$\theta = \tan^{-1} \left(\frac{a_y}{a_x} \right) = 13^\circ$$



i.e. a makes 193° with +ve x -axis.

Now, $x = \int v_x dt = \int 2 \cos t dt = 2 \sin t$

$$y = \int v_y dt = \int \sin t dt = -\cos t$$

We have,

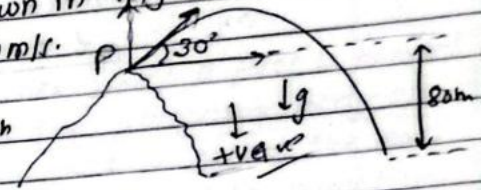
$$\sin t = \frac{x}{2}, \cos t = -y$$

Thus, $\sin^2 t + \cos^2 t = \frac{x^2}{4} + y^2 = 1$

$$\Rightarrow \frac{x^2}{4} + y^2 = 1 \rightarrow \text{Path is elliptical.}$$

9

A bullet is fired at upward at an angle of 30° to a horizontal from a point 'P' on a hill. And it strikes a target which is 80m lower than the level of projection as shown in figure. The initial velocity of bullet is 100 m/s . Calculate



(a) The max height to which bullet will rise above the horizontal

(b) Actual velocity with which it will strike the target.
 (c) Total time required for the flight of the bullet. (Neglect Air resistance)

→ Sol:

$$S = u_y t + \frac{1}{2} g t^2$$

or, $80 = - (100 \sin 30^\circ) t + \frac{1}{2} g t^2$

At max height, $v_y = 0$

(a) $v_y^2 = u_y^2 + 2 g s$

or, $0^2 = (100 \sin 30^\circ)^2 + 2 \times (-9.8) \times s$

or, $H_{\text{max}} = 127.29 \text{ m} \rightarrow \text{above horizontal}$
 $\hookrightarrow \text{from top of cliff}$

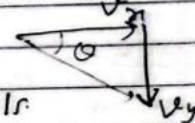
Above ground, $H = 127.29 + 80 = 207.29 \text{ m}$.

(b) Let, it strike with velocity v'

$$v_y'^2 = u_y^2 + 2 g s$$

or, $v_y'^2 = (-100 \sin 30^\circ)^2 + 2 \times 9.82 \times 80$

or, $v_y' = 63.81 \text{ m/s}$



$$v_x = u_x = 100 \cos 30^\circ = 86.6 \text{ m/s}$$

Thus, $|v'| = \sqrt{v_x^2 + v_y^2} = 107.57 \text{ m/s}$

Direction of velocity = $\tan^{-1} \left(\frac{v_y}{v_x} \right) = 36.88^\circ$ with v_x

Projectile motion

↳ The motion of projectile can be replaced by two independent rectilinear motion.

i.e. uniformly accelerated ^{rectilinear} motion { vertical }
& uniformly ^{rectilinear} motion ($a=0$) { horizontal }

* Illustrate

→ For a projectile motion, horizontal acceleration (a_x) = 0

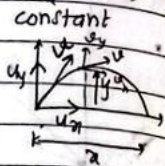
i.e. uniform rectilinear motion

Thus, horizontal velocity of particle remains same throughout the motion $v_x = u_x = \text{constant}$

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From eqs of motion, $v_y^2 = u_y^2 + 2g\delta$



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where x & y are expressed in mm and t in second.

Determine velocity and acceleration when $t=1$ second. Also, Determine the path of the particle

→ Soln:-

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$$\ddot{y} = a_y = \frac{d^2y}{dt^2} = -100\pi^2 \cos 2\pi t$$

path of the particle is parabolic
↳ $x^2 + 200y = 5000$
 $100\pi^2$

Magnitude

$$\text{At } t=1, |\vec{v}| = \sqrt{v_x^2 + v_y^2} = \sqrt{(100\pi \cos \pi)^2 + (50\pi \sin 2\pi)^2}$$

$$= 100\pi \text{ mm/s}$$

Direction,

$$\theta = \tan^{-1} \left(\frac{v_y}{v_x} \right) = 0^\circ$$

Magnitude,

$$\text{At } t=1, |\vec{a}| = \sqrt{a_x^2 + a_y^2} = \sqrt{(-100\pi^2 \sin \pi)^2 + (100\pi^2 \cos 2\pi)^2}$$

$$= 100\pi^2 \text{ mm/s}^2$$

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$$y = \frac{25}{2} \left\{ 1 - 2 \frac{x^2}{100^2} \right\}$$

$$\text{or } y = \frac{25(10000 - x^2)}{10000}$$

$$\text{or } 200y = 25 - \frac{25x^2}{5000}$$

$$\text{or } 200y = 5000 - x^2$$

$$\text{or } x^2 + 200y = 5000 \rightarrow \text{parabolic}$$

14

$$v_x = 2 \cos t$$

$$v_y = \sin t$$

Initially it is known that both x & y coordinate are 0. And determine the path of particle & total acceleration at 2 sec. & $\rightarrow$

$\rightarrow$ Solⁿ:

$$v_x = 2 \cos t$$

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Now,

$$a_x = \frac{dv_x}{dt} = -2 \sin t$$

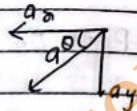
$$a_y = \frac{dv_y}{dt} = \cos t$$

At $t = 2 \text{ sec}$, $a_x = -1.82 \text{ m/s}^2$

$$a_y = -0.42 \text{ m/s}^2$$

$$\text{Thus, } |\vec{a}| = \sqrt{a_x^2 + a_y^2} = 1.87 \text{ m/s}^2$$

$$\theta = \tan^{-1} \left(\frac{a_y}{a_x} \right) = 13^\circ$$



i.e. $\vec{a}$ makes 193° with +ve x axis.

Now, $x = \int v_x dt = \int 2 \cos t dt = 2 \sin t$

$$y = \int v_y dt = \int \sin t dt = -\cos t$$

We have,

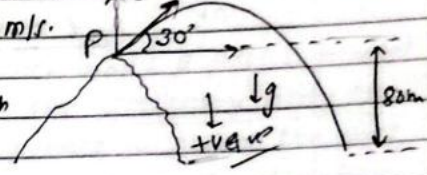
$$\sin t = \frac{x}{2}, \cos t = -y$$

$$\text{Thus, } \sin^2 t + \cos^2 t = \frac{x^2}{4} + y^2 = 1$$

$$\Rightarrow \frac{x^2}{4} + y^2 = 1 \rightarrow \text{Path is elliptical.}$$

15

A bullet is fired upward at an angle of 30° to a horizontal from a point 'P' on a hill. And it strikes a target which is 80m lower than the level of projection as shown in figure. The initial velocity of bullet is 100 m/s. calculate



- The max^m height to which bullet will rise above the horizontal
- Actual velocity with which it will strike the target
- Total time required for the flight of the bullet. (Neglect Air resistance)

$\rightarrow$ Solⁿ:

$$s = u_y t + \frac{1}{2} g t^2$$

$$\text{or, } 80 = - (100 \sin 30^\circ) t$$

At max^m height, $v_y = 0$

$$(a) \quad v_y^2 = u_y^2 + 2 g s$$

$$\text{or, } 0^2 = (100 \sin 30^\circ)^2 + 2 \times (-9.8) \times s$$

$$\text{or, } H_{\text{max}} = 127.29 \text{ m} \rightarrow \text{above horizontal}$$

from top of cliff

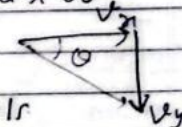
$$\text{Above ground, } H = 127.29 + 80 = 207.29 \text{ m.}$$

- Let, it strike with velocity v'

$$v'^2 = u_y^2 + 2 g s$$

$$\text{or, } v'^2 = (-100 \sin 30^\circ)^2 + 2 \times 9.8 \times 80$$

$$\text{or, } v' = 63.81 \text{ m/s}$$



$$v_x = u_x = 100 \cos 30^\circ = 86.6 \text{ m/s}$$

$$\text{Thus, } |\vec{v}| = \sqrt{v_x^2 + v_y^2} = 107.57 \text{ m/s}$$

$$\text{Direction of velocity} = \tan^{-1} \left(\frac{v_y}{v_x} \right) = 36.88^\circ \text{ with } v_x$$

$-63.81 = +$

↑ +ve

(c) For time of flight

$v_y = u_y + g t$

or $-63.81 = -100 \sin 30^\circ + 9.82 t$

or $t = 11.59 \text{ sec}$

Friction

classmate
Date _____
Page _____
 $\mu_s > \mu_k > \mu_r$

Define Friction, laws of friction, coefficients of friction, angle of friction, angle of repose, push & pull.

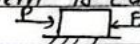
Friction:-

When a body slides over another body, a force is exerted at a surface of contact by the stationary body on a moving body. The resisting force is called the force of friction and always acts in opposite direction to the direction in which body tends to move.

(i) Static friction: Frictional force exerted by a body at rest.

Frictional force that develops b/w mating surface when subjected to external force but there is no relative motion b/w them is called static friction.

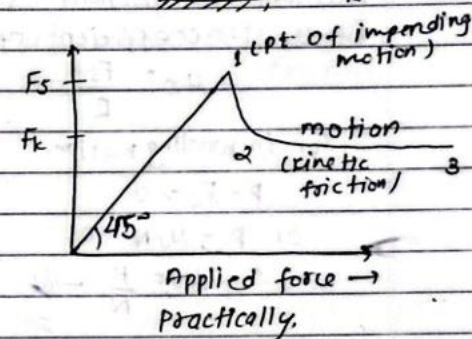
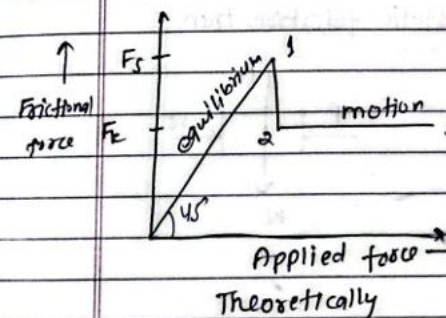
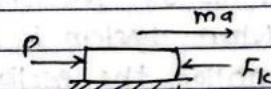
$F_s = P \rightarrow \text{no motion.}$



(ii) Dynamic friction

Frictional force that develops between mating surfaces when subjected to external forces and there is relative motion between them is called dynamic/kinetic friction.

$F_k < P$



Dry friction → Force that opposes sliding across another solid surface. (classmate)
 (i) Static (blw non-moving surfaces)
 (ii) Kinetic (blw moving surfaces)

Laws of friction (Static & Dynamic)

- Friction acts tangential to the surface of contact and is in a direction opposite to that in which motion is to take place.
- Frictional force is max^m at instant of impending motion (diminishing friction condⁿ)
- The ratio of limiting friction (F_s) to normal reaction (R) always constant.

$$\frac{F_s}{R} = \mu_s (\text{Constant})$$

This ratio drops to a slightly lower value when motion starts

- Limiting friction is independent of area and shape of contact surface.
- Frictional force depends on roughness of surface. (nature)
- At low velocity, kinetic friction is constant. But, at higher speed, there will be slight reduction in friction. (kinetic)

Coefficients of friction

The frictional force is proportional to normal reaction i.e. $F \propto R$. The ratio F/R is called coefficient of friction. So, coefficient of friction is defined as ratio of frictional force to normal reaction.

When system is in the state of impending motion, the frictional force has max^m value F_s .

If μ_s is coefficient of static friction, then

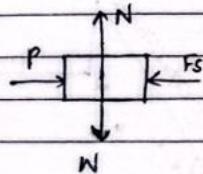
$$\mu_s = \frac{F_s}{R}$$

For impending motion,

$$P - F_s = 0$$

$$\text{or, } P = \mu_s N = 0$$

$$\therefore \mu_s = \frac{P}{N} \quad \text{---(i)}$$

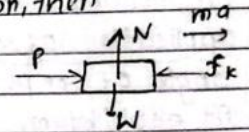


* $F_f \propto$ Normal force

- dependent on type and condition of contact surface
- Independent of contact area.

When motion starts, max^m force of friction falls to some lower value called kinetic friction F_k . If μ_k is coefficient of kinetic friction, then

$$\mu_k = \frac{F_k}{R}$$



For motion,

$$P - F_k = ma \quad \left\{ \begin{array}{l} m \rightarrow \text{mass} \\ a \rightarrow \text{acceleration} \end{array} \right.$$

$$\text{or, } P - \mu_k N = ma$$

$$\text{or, } \mu_k = \frac{P - ma}{N} \quad \text{---(ii)}$$

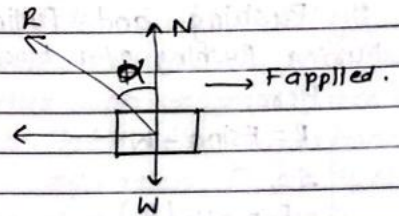
Since, $a > 0$, from eqⁿ (i) and (ii), $\mu_s > \mu_k$.

Thus

Coefficient of static friction is always greater than kinetic friction.

Angle of friction (α)

It is the angle made by resultant of Normal reaction and limiting frictional force with normal reaction.



$N \rightarrow$ Normal reaction

$F_s \rightarrow$ Limiting friction

$$R = \sqrt{N^2 + F_s^2}$$

$$\therefore \tan \alpha = \frac{F_s}{N} = \mu_s, \text{ where } \alpha \text{ is angle of friction}$$

- * Frictional forces are in reverse direction.
- * Bearings, power screws, flow of fluids in pipes etc.
- * Propulsion of aircraft through atmosphere etc.
- * Friction results in loss of energy which is dissipated in form of heat.
- * Causes wear and tear of machine parts.

Angle of repose (θ)

The minimum angle of inclination for which the body on inclined surface is in the verge of motion without application of any external force is known as angle of repose.

At equilibrium,

$$F_s = mg \sin \theta$$

$$R = mg \cos \theta$$

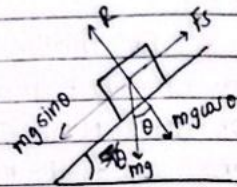
Thus,

$$\tan \theta = \frac{F_s}{R} = \mu_s \quad \text{--- (i)}$$

Thus, from (i) & (ii),

$$\theta = \alpha$$

Angle of repose = Angle of friction



Pushing and Pulling the Block.

During pushing of a block,

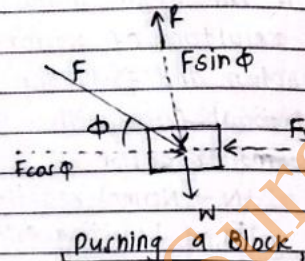
Here,

$$R = F \sin \phi + W$$

So,

$$F_s = \mu_s (R)$$

$$F_s = \mu_s (F \sin \phi + W) \quad \text{--- (i)}$$



Let, pulling and pushing force has same magnitude 'F' and makes same angle 'phi'

In pulling,

$$R + F \sin \phi = W$$

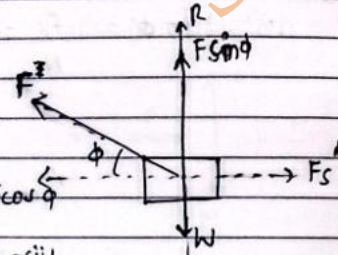
$$\text{or, } R = W - F \sin \phi$$

$$\text{So, } F_s' = \mu_s (R)$$

$$F_s' = \mu_s (W - F \sin \phi) \quad \text{--- (ii)}$$

$F_s' < F_s$ i.e. More opposing force during pushing.

Thus, pulling is easier than pushing.



- * Frictional forces are Desired in some cases.
- * Brakes, clutches
- * Walking depends on friction b/w shoe & ground.

Four different situations that occur when rigid body is in contact with horizontal surface.

(1) The force applied to the body, there is no friction.

(2) The applied force tend to move the body along the surface of contact but are not large enough to set in motion. The frictional force 'F' which has developed may be found by solving equations of equilibrium for the body. Since, there is no evidence that 'F' has reached its maximum value, the equation $F_{max} = \mu_s N$ cannot be used to determine the frictional force.

(3) The applied force are such that the body is just about to slide, the motion is impending. The frictional force 'F' has reached its maximum value 'F_{max}' and together with normal force 'N' balance the applied force. Both the equations of equilibrium and the equation $F_{max} = \mu_s N$ may be used. The frictional force has a sense opposite to sense of impending motion.

(4) The body is sliding under the action of applied force. and The equation $F_k = \mu_k N$ may be used.

N = P + W (a) NO Friction (Force applied no friction)	N = P + W (b) NO Motion	N = P + W (c) Motion Impending	N = P + W (d) Motion.
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Dynamics

to find at any point the normal component of velocity along the path

contd.

Tangential and normal components

1. Consider a particle moving along a path

Let us suppose, $\hat{e}_t$ and $\hat{e}_n$ are unit vectors of tangential & normal at point P

And, $\hat{e}_t$ and $\hat{e}_n$ are unit vectors of tangential & normal component when particle is at point P

We know,

Velocity is tangent to path

$$\vec{v} = v \hat{e}_t + 0 \hat{e}_n$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d(v \hat{e}_t)}{dt} = \dot{v} \hat{e}_t + v \frac{d\hat{e}_t}{dt}$$

$$= \dot{v} \hat{e}_t + v \frac{d\theta}{dt} \cdot \frac{d\hat{e}_t}{d\theta} \cdot \frac{d\theta}{dt}$$

$$= \dot{v} \hat{e}_t + v \left(\frac{1}{r} \right) (v) \frac{d\theta}{dt}$$

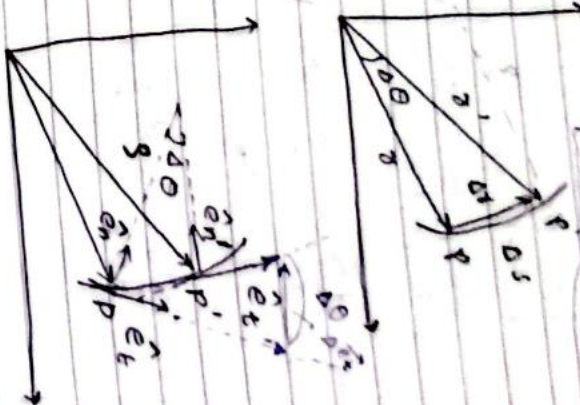
$$\vec{a} = \dot{v} \hat{e}_t + \frac{v^2}{r} \hat{e}_n$$

Note: $\frac{d\theta}{ds} = \lim_{\Delta s \rightarrow 0} \frac{\Delta \theta}{\Delta s} = \frac{1}{r}$

$$\frac{d\hat{e}_t}{d\theta} = \lim_{\Delta \theta \rightarrow 0} \frac{\Delta \hat{e}_t}{\Delta \theta} = \lim_{\Delta \theta \rightarrow 0} \frac{(1 \times \Delta \theta) \hat{e}_n}{\Delta \theta} = \hat{e}_n$$

$$|\Delta \hat{e}_t| = \Delta \theta \times \hat{e}_n$$

both normal & tangential comp



Condition of sliding and overturning

As the magnitude of frictional force 'F' increases from 0 to F_{max} , the point of application 'P' of the resultant 'N' of the normal force of contact moves to the right so that, the couple formed respectively by (P & F) and (W & N) remain balanced by (P & F) and 'B' before 'F' reaches its maximum value (F_{max}), the block will overturn about 'B' before it starts sliding i.e. $F_{lim} > F$ (actual calculated frictional force)

→ If $F_{lim} > F$ (actual calculated frictional force) the block will overturn over the plane instead of sliding.



- $a = \frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt} = v \frac{dv}{dx}$
- $v = \frac{dx}{dt} \Rightarrow dx = v dt$
- Uniform rectilinear motion: $\frac{velocity}{time} = \text{const}$
- Uniform accel: rectilinear motion: $a = c$
- Uniform vel & position: velocity varies linearly

- # Conclusion:-
1. If velocity increase, a_t will be same sense.
 2. If velocity decrease, a_t will be opposite sense of velocity
 3. If speed constant, $a_t = 0$.
 4. $\vec{a}_n$ always directed towards centre of curvature.
 5. If $S = \infty$ (path straight line)
 $\Rightarrow a_n = 0 \Rightarrow$ only a_t occurs.

Tangential acclⁿ is due to change in magnitude of velocity.
i.e. tangential acclⁿ is defined by resultant velocity
AS, velocity vector is always tangent to path.

Normal Acclⁿ is due to change in direction of velocity vector.

Radial & Transverse component

→ radial: along the radius {pole to point}

→ Both velocity as well as accelⁿ has both radial and transverse component.

→ Transverse: $\perp$ to radial component

$$\rightarrow \vec{v} = \dot{r} \hat{e}_r + r \dot{\theta} \hat{e}_\theta \quad \left\{ \begin{array}{l} \dot{r} = \frac{dr}{dt} \\ \dot{\theta} = \frac{d\theta}{dt} \end{array} \right.$$

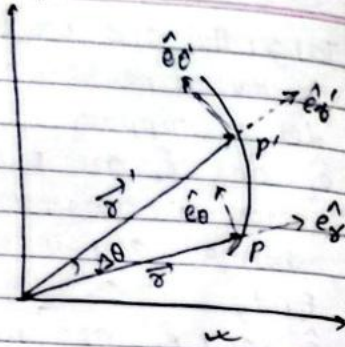
$$\rightarrow \vec{a} = (\ddot{r} - r \dot{\theta}^2) \hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta}) \hat{e}_\theta \quad \left\{ \begin{array}{l} \dot{r} = \frac{dr}{dt} \\ \dot{\theta} = \frac{d\theta}{dt} \end{array} \right.$$

→ Note: For particle moving along a circular path with its centre as origin 'O' ($r = \text{constant}$) $\Rightarrow \dot{r} = 0$ & $\ddot{r} = 0$

$$\text{So, } \vec{v} = r \dot{\theta} \hat{e}_\theta \quad | \quad \vec{a} = -r \dot{\theta}^2 \hat{e}_r + r \ddot{\theta} \hat{e}_\theta$$

Radial and Transverse component

→ $\hat{e}_r$ & $\hat{e}_\theta$; $\hat{e}_r$ & $\hat{e}_\theta$ be unit vectors along radial and transverse components at P & P' respectively.



$$\rightarrow \frac{d\hat{e}_r}{d\theta} = \lim_{\Delta\theta \rightarrow 0} \frac{\Delta\hat{e}_r}{\Delta\theta}$$

$$= \lim_{\Delta\theta \rightarrow 0} \frac{(1 \times \Delta\theta) \hat{e}_\theta}{\Delta\theta} = \hat{e}_\theta$$

$$\begin{aligned} \frac{d\hat{e}_\theta}{d\theta} &= \frac{d\hat{e}_\theta}{d\theta} \times \frac{d\theta}{dt} = \lim_{\Delta\theta \rightarrow 0} \frac{\Delta\hat{e}_\theta}{\Delta\theta} \cdot \frac{d\theta}{dt} \\ &= \lim_{\Delta\theta \rightarrow 0} \frac{(1 \times \Delta\theta) (-\hat{e}_r)}{\Delta\theta} \cdot \dot{\theta} \\ &= -\hat{e}_r \dot{\theta} \end{aligned}$$

$$\rightarrow \vec{r} = r\hat{e}_r$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d(r\hat{e}_r)}{dt} = \dot{r}\hat{e}_r + r\dot{\hat{e}_r} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

$$\begin{aligned} \vec{a} &= \frac{d\vec{v}}{dt} = \frac{d(\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta)}{dt} \\ &= \ddot{r}\hat{e}_r + \dot{r}\frac{d\hat{e}_r}{dt} + \dot{\theta}\frac{d(r\hat{e}_\theta)}{dt} + r\frac{d\dot{\theta}\hat{e}_\theta}{dt} \\ &= \ddot{r}\hat{e}_r + \dot{r}\frac{d\hat{e}_r}{d\theta} \frac{d\theta}{dt} + \dot{\theta}(\dot{\theta}r\hat{e}_\theta + r\dot{\hat{e}_\theta}) + r\dot{\theta}\frac{d\hat{e}_\theta}{dt} \\ &= \ddot{r}\hat{e}_r + \dot{r}\hat{e}_\theta \dot{\theta} - r\dot{\theta}^2\hat{e}_r + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}\dot{\hat{e}_\theta} \end{aligned}$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$

$$\frac{d\hat{e}_r}{dt} = \frac{d\hat{e}_r}{d\theta} \frac{d\theta}{dt} = \hat{e}_\theta \dot{\theta}$$

The motion of particle is defined by relation position vector: $\vec{r} = 3t^2\hat{i} + 4t^3\hat{j} + 5t^4\hat{k}$ where, $\vec{r}$ is in 'm' & t is in seconds.

At the instant when $t = 4$ sec, Find normal & tangential component of acceleration & radius of curvature.

→ Sol:

$$\vec{r} = 3t^2\hat{i} + 4t^3\hat{j} + 5t^4\hat{k}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = 6t\hat{i} + 12t^2\hat{j} + 20t^3\hat{k}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = 6\hat{i} + 24t\hat{j} + 60t^2\hat{k}$$

$$v_y = \sqrt{(6 \times 4)^2 + (12 \times 4^2)^2 + (20 \times 4^3)^2} = 1294.54 \text{ m/s}$$

$$a_y = \sqrt{6^2 + (24 \times 4)^2 + (60 \times 4^2)^2} = 964.81 \text{ m/s}^2$$

Tangential component of acceleration (a_t) = $\dot{v}$
At $t = 4$ sec } $a_t = 964.81 \text{ m/s}^2$

total accelⁿ

$$S = \left[1 + \left(\frac{dv}{dt} \right)^2 \right]^{3/2}$$

At $t = 4$ sec

$$a \cos \theta = \frac{\vec{a} \cdot \vec{v}}{|\vec{v}|} = \frac{144 + 288 + 2240}{1294.54}$$

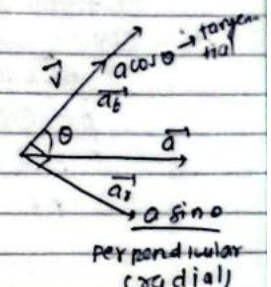
Or Tangential component $a \cos \theta = 2.064 \text{ m/s}^2$

$$\vec{a} = 6\hat{i} + 96\hat{j} + 960\hat{k} \Rightarrow |\vec{a}| = 964.81 \text{ m/s}^2$$

$$\vec{a} = \vec{a}_t + \vec{a}_n$$

$$|\vec{a}| = \sqrt{a_t^2 + a_n^2}$$

$$\Rightarrow a_n = 964.81 \text{ m/s}^2 \text{ (normal)}$$



Now, $a_n = \frac{v^2}{r}$

or, $964.81 = \frac{(1294.54)^2}{r}$

or, $r = 1736.96m$

Q. The rotation of a 0.9m arm OA about O is defined by the relation $\theta = 0.15t^2$, where θ is in radian and t in seconds.

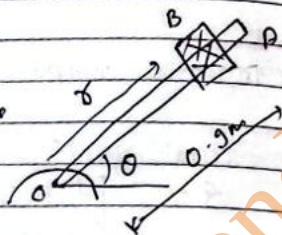
Collar B slides along the arm in such a way that its distance from O is

$r = (0.9 - 0.12t^2)$

After the arm OA has rotated through 30° . Determine

(i) total velocity

(ii) total acceleration of collar



→ Solⁿ:

$\theta = 0.15t^2$	$r = 0.9 - 0.12t^2$
$\dot{\theta} = 0.3t$	$\dot{r} = -0.24t$
$\ddot{\theta} = 0.3$	$\ddot{r} = -0.24$

At $\theta = 30^\circ = \pi/6$

or $\pi/6 = 0.15t^2$

or $t = 1.868 \text{ sec.}$

At $t = 1.868 \text{ sec.}$

$\theta = 0.523$	$r = 0.48$
$\dot{\theta} = 0.5604$	$\dot{r} = -0.448$
$\ddot{\theta} = 0.3$	$\ddot{r} = -0.24$

Now, (i) $\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$

At $\theta = 30^\circ$

$\vec{v} = -0.448\hat{e}_r + 0.269\hat{e}_\theta \text{ m/s}$

$|\vec{v}| = 0.523 \text{ m/s}$

$\phi = \tan^{-1} \left(\frac{-0.448}{0.269} \right) = 59^\circ \text{ (below } \hat{e}_\theta \text{ dir.)}$

(ii) $\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$

At $\theta = 30^\circ$

$\vec{a} = -0.39\hat{e}_r - 0.358\hat{e}_\theta \text{ m/s}^2$

$|\vec{a}| = 0.529 \text{ m/s}^2$

$\phi = \tan^{-1} \left(\frac{-0.358}{-0.39} \right) = 42.55^\circ$

Kinetics of Particle

Newton's 2nd law:

The rate

$$F = m\vec{a} \rightarrow \text{Effect of Newton's 2nd law}$$

$$\vec{F} = \frac{d(m\vec{v})}{dt} \rightarrow \text{2nd law}$$

Linear momentum & rate of change; Eqⁿ of motion and dynamic eqⁿ, angular momentum & rate of change

$$\vec{F} dt = d(m\vec{v})$$

$$\text{Impulse} = \int \vec{F} dt = m\vec{v}_2 - m\vec{v}_1$$

time t_1 to t_2 $\rightarrow$ change in momentum during that interval.

$$\vec{F} = m\vec{a}$$

$$[\vec{F} - m\vec{a} = 0] \Rightarrow \text{Dynamic eqⁿ .}$$

When particle is considered to be in equilibrium and under the given force and inertia vector,

it is said to be dynamic equilibrium.

$\rightarrow$ It is also called D'Alembert's principle.

Angular momentum & rate

Newton's 2nd Law:

"The force acting on a body is equal to the rate of change of ^{linear} momentum of that body."

$$\vec{F} = \frac{d\vec{P}}{dt}$$

$$\text{or, } \vec{F} = \frac{d(m\vec{v})}{dt}$$

$$\text{or, } \vec{F} = m\vec{a}$$

It can also be stated as, "If the resultant force on a particle is not zero, the particle will have an acceleration proportional to the magnitude of resultant and in the direction of this resultant force."

$$\text{if } \vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k} \text{ \& }$$

$$\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$$

Then,

$$F_x\vec{i} + F_y\vec{j} + F_z\vec{k} = m(a_x\vec{i} + a_y\vec{j} + a_z\vec{k})$$

$$\text{So, } F_x = ma_x, F_y = ma_y, F_z = ma_z$$

$\rightarrow$ Newton's law of motion defines the relationship between force, mass & acceleration.

Also, from Newton's 2nd law,

$$\vec{F} = \frac{d(m\vec{v})}{dt}$$

$$\text{or, } \vec{F} dt = d(m\vec{v})$$

Integrating,

$$\int_{t_1}^{t_2} \vec{F} dt = \int_{\vec{v}_1}^{\vec{v}_2} d(m\vec{v})$$

$$\text{or, } \int_{t_1}^{t_2} \vec{F} dt = m(\vec{v}_2 - \vec{v}_1)$$

$$\text{or, } \boxed{m\vec{v}_2 = \int_{t_1}^{t_2} \vec{F} dt + m\vec{v}_1} \quad \text{--- (2)}$$

where, $\int_{t_1}^{t_2} \vec{F} dt = \vec{I}_{1-2}$ is called linear impulse or simply impulse.
Eqn (2) is also called impulse-momentum principle.

Linear momentum

The vector $m\vec{v}$ is called linear momentum or simply momentum. It has same direction as velocity and magnitude equal to product of mass and velocity. It is denoted by $\vec{L}$ (or $\vec{Q}$ or $\vec{P}$).

$$\vec{L} = m\vec{v}$$

→ Rate of change of linear momentum is force.

Equation of motion and Dynamic Equilibrium.

Equation of motion

Considering a particle of mass (m) acted upon by several forces, then from Newton's 2nd law,

$$\sum \vec{F} = m\vec{a}$$

(a) Rectangular component

$$\sum F_x = ma_x, \sum F_y = ma_y, \sum F_z = ma_z$$

(b) Tangential and normal components

$$\rightarrow \sum F_t = ma_t, \sum F_t = m \frac{dv}{dt}$$

$$\rightarrow \sum F_n = ma_n, \sum F_n = m \frac{v^2}{r}$$

(c) Radial and transverse component

$$\sum F_r = ma_r$$

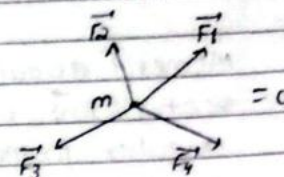
$$\sum F_\theta = ma_\theta$$

Dynamic Equilibrium,

From Newton's second's law.

$$\sum \vec{F} = m\vec{a}$$

$$\text{or } \sum \vec{F} - m\vec{a} = 0 \quad \text{--- (1)}$$



The term $-m\vec{a}$ is inertia force or reverse effective force which has magnitude ma and direction opposite to that of acceleration.

Thus, when the particle is in equilibrium, under the given forces & the inertia vector $-m\vec{a}$, then it is said to be in dynamic equilibrium.

Inertia vector measures the resistance that particle offer when we try to set them in motion or when we try to change the condition of their motion.

Angular momentum and rate of change

Angular momentum

The moment of linear momentum is called angular momentum, about a reference point (O).

$$\text{Angular momentum } (\vec{H}_O) = \vec{r} \times (m\vec{v})$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$$

Then

$$\vec{H}_O = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ mv_x & mv_y & mv_z \end{vmatrix}$$

Mass centre

Moment due to entire mass = Sum of moments
due to individual
masses

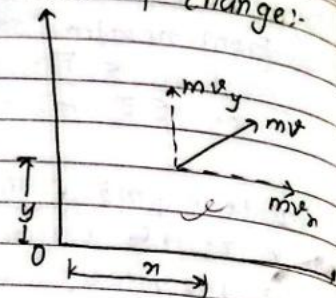
$$\vec{r}_c * M = \sum \vec{r}_i * m_i$$

$$\therefore \boxed{\vec{r}_c = \frac{\sum \vec{r}_i * m_i}{M}}$$

→ mass
centre of
particles

Angular momentum & rate of change:

Moment about O of vector $m\vec{v}$ is called angular momentum of particle about 'O' at that instant.



$$H_O = x(mv_y) - y(mv_x)$$

Rate of change of angular momentum

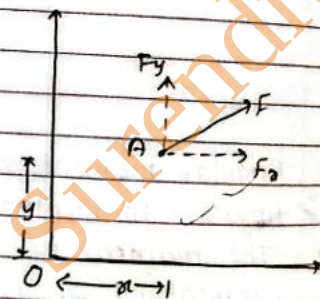
$$\frac{dH_O}{dt} = \frac{d}{dt} [xmv_y - ymv_x]$$

$$= x \frac{dmv_y}{dt} - y \frac{dmv_x}{dt}$$

$$= xma_y - yma_x$$

$$= xF_y - yF_x$$

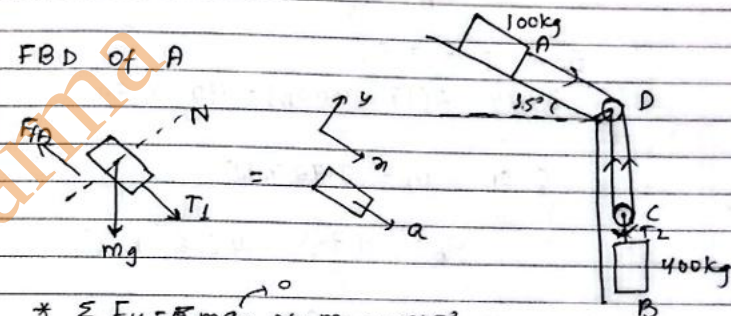
Rate of change of angular momentum of particle about any point at any instant is equal to the moment of force $\vec{F}$ acting on the particle about the same point.



Q. Two blocks shown in figure are attached with frictionless pulley with no mass. Coefficient of static and kinetic friction b/w block A and inclined plane are 0.35 and 0.25. Determine acceleration of each block and tension in each cord.

→ Soln:

FBD of A



$$\sum F_y = 0 \Rightarrow N = m_A g \cos 15^\circ$$

$$\sum F_x = m_A a_A$$

$$T_1 + m_A g \sin 15^\circ - F_{fA} = m_A a_A$$

$$\text{or, } T_1 + 100 \times 9.81 \times \sin 15^\circ - \mu_k (m_A g \cos 15^\circ) = 100 a_A$$

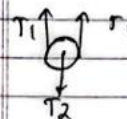
$$T_1 - 100 a_A = 17 \quad \text{--- (1)}$$

FBD of B

$$\sum F_y = 0$$

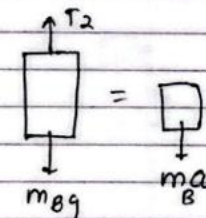
$$\text{or, } m_B g - T_2 = m_B a_B$$

$$\text{or, } 3924 - T_2 = 400 a_B \quad \text{--- (2)}$$



$$T_2 - 2T_1 = 0$$

$$\Rightarrow T_2 = 2T_1 \quad \text{--- (3)}$$



When block A moves through s_A distance the block B moves $\frac{s_A}{2}$ distance.

$$\therefore s_B = \frac{s_A}{2}$$

Differentiating,

$$a_B = \frac{a_A}{2} \quad \text{--- (4)}$$

From (1), (2), (3), (4),

$$3924 - 2(17 + 100a_A) = 400 \times \frac{a_A}{2}$$

$$\therefore a_A = 9.725 \text{ m/s}^2$$

$$a_B = \frac{9.725}{2} = 4.8625 \text{ m/s}^2$$

$$T_1 = 17 + 100a_A = 989.5 \text{ N}$$

$$T_2 = 2T_1 = 1979 \text{ N}$$

① Relative motion
② Dependent motion

(Q) A collar of 5 kg mass slides without friction along a vertical rod as shown. The spring attached to the collar has a spring constant of 250 N/m and its undeformed length is 15 cm. The collar is released from rest at position A and slides 20 cm downward to position B. What will be the acceleration of collar at position B?

→ Solⁿ.

m_{collar}

At position B,

Stretched length (L)

$$= 25 \text{ cm}$$

Spring force (F_s) = $k(\Delta x)$

$$= 250(25 - 15) \times 10^{-2}$$

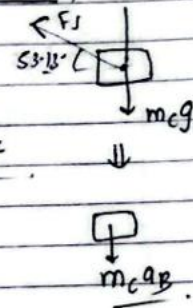
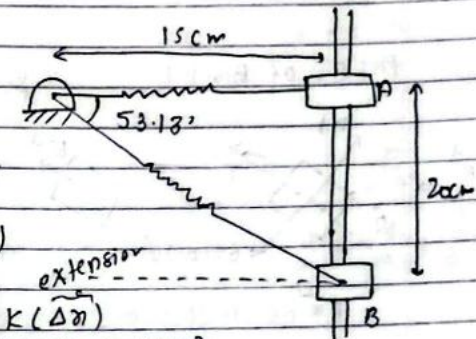
$$= 2500$$

$$= 25 \text{ N}$$

$$\sum F_y = m a_B$$

$$m_{\text{collar}} g - 25 \sin(53.13^\circ) = m_{\text{collar}} a_B$$

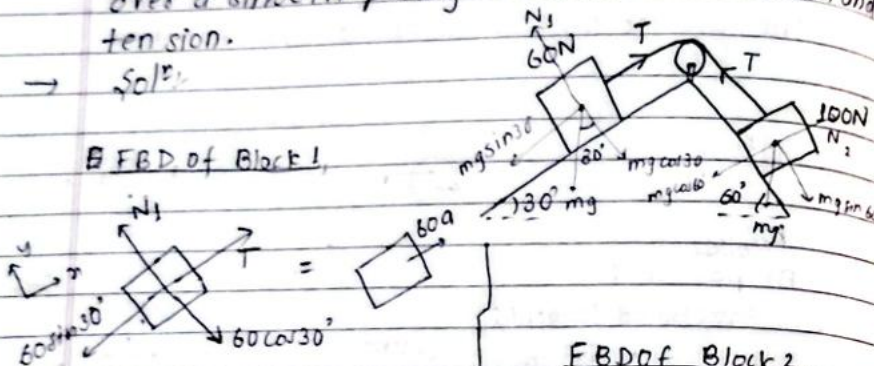
$$\Rightarrow a_B = 8.99 \text{ m/s}^2 \quad \underline{5.82 \text{ m/s}^2}$$



- (Q) Two smooth inclined plane which inclination with horizontal are 60° and 30° are placed back to back. Two bodies of weight 100N and 60N are placed on them and are connected by light inextensible string passing over a smooth pulley. Calculate acceleration and tension.

→ Solⁿ:

FBD of Block 1,

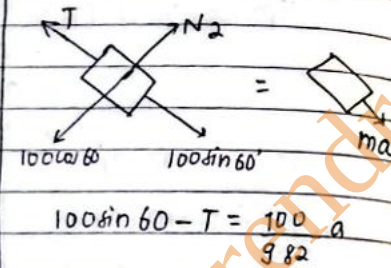


$$T - 60 \sin 30^\circ = \frac{60}{9.82} \times a$$

$$\text{or } T = 6.11a + 30 \quad \text{--- (1)}$$

$$N_1 = 51.96\text{N}$$

FBD of Block 2,



$$100 \sin 60 - T = \frac{100}{9.82} a$$

$$\text{or } T = 86.6 - 10.18a \quad \text{--- (2)}$$

From (1) & (2),

$$6.11a + 30 = 86.6 - 10.18a$$

$$\Rightarrow a = 3.47 \text{ m/s}^2$$

$$T = 6.11 \times 3.47 + 30 = 51.23\text{N}$$

bob of a 3m

- (Q) A pendulum describes an arc of a circle in a vertical plane. If the tension in the chord is 2 times the weight of the bob for the position when it is displaced through 30° from its mean position. Find the velocity and acceleration.

Applying equations of eq^m at give position.

$$\Sigma F_y = ma_n$$

$$\text{or } 2mg - mg \cos 30^\circ = ma_n$$

$$\text{or } a_n = 11.135 \text{ m/s}^2$$

$$\Sigma F_t = ma_t$$

$$mg \sin 30^\circ = ma_t$$

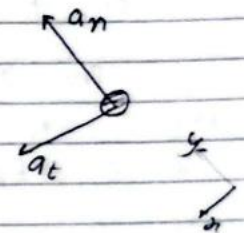
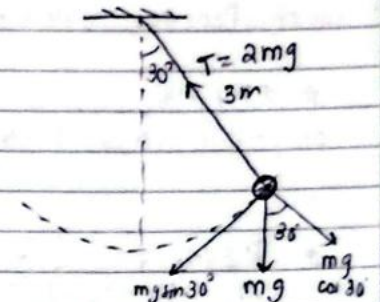
$$\Rightarrow a_t = 4.91 \text{ m/s}^2$$

$$\begin{aligned} \text{Total acceleration (a)} &= \sqrt{a_n^2 + a_t^2} \\ &= \sqrt{(11.135)^2 + (4.91)^2} \\ &= 12.169 \text{ m/s}^2 \end{aligned}$$

Also,

$$a_n = \frac{v^2}{r} \rightarrow 3$$

$$\Rightarrow v = 5.78 \text{ m/s} \quad \text{velocity (tangential)}$$



Friction

- (Q) A homogeneous block has a mass of 'm' kg & the coefficient of friction between the block & the plane is 0.4. If the force 'P' increases gradually until motion ensues (start) will the block slide or overturn & for what values of 'P'.

→ Solⁿ:

* $\sum F_x = 0$

or, $P \cos 20^\circ - mg \sin 20^\circ - F = 0 \quad \text{--- (1)}$

* $\sum F_y = 0$

or, $-P \sin 20^\circ - mg \cos 20^\circ + N = 0 \quad \text{--- (2)}$

or,

* $\sum M_A = 0$ (2 +ve)

or, $P \cos 20^\circ \times 125 - mg \cos 20^\circ \times 50 - mg \sin 20^\circ \times 125 = 0$

or, $117.46P - 460.92m - 419.4m = 0$

or, $P = 7.495m \quad \text{--- (1)}$

from (2)

$N = 11.7818m \quad \text{--- (1)}$

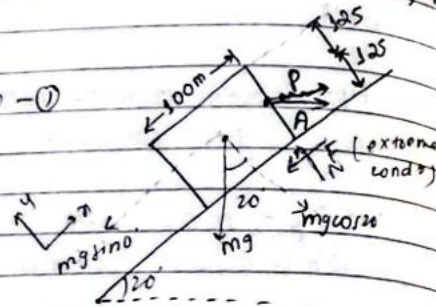
$F_{lim} = \mu N = 0.4 \times 11.7818m = 4.7127m$

From (1)

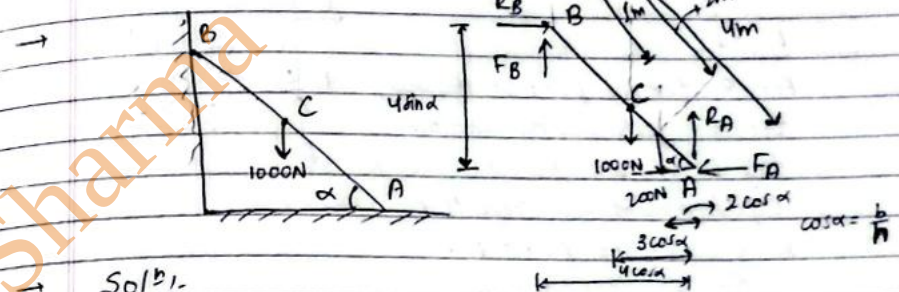
$F = 2.175m$

$F < F_{lim}$ } So, the block will ~~slide~~ ^{overturn}

$P = 7.495m$



- (Q) A ladder 4m long as shown is supported by a horizontal floor and a vertical wall. The coefficient of friction at the wall is 0.25 and at the floor is 0.5. The weight of the ladder is 200N. A ladder supports a vertical load of 1000N at C. Determine reaction at A and B and compute least value of ' α ' at which the ladder may be placed without slipping to right.



Solⁿ:

Using equations of equilibrium for rigid body:

$\sum F_x = 0$

$R_B = F_A \quad \text{--- (1)}$

$F_B = \mu_w R_B = 0.25 R_B$

$\Rightarrow R_B - 0.5 R_A = 0 \quad \text{or} \quad F_A = \mu_f R_A = 0.5 R_A$

$\sum F_y = 0$

$F_B + R_A = 200 \quad \text{--- (1)}$

$\Rightarrow 0.25 R_B + R_A = 200$

* $\sum M_A = 0$ (2 +ve)

or, $200 \times 2 \cos \alpha + 1000 \times 3 \cos \alpha$

$- F_B \times 4 \cos \alpha - R_B \times 4 \sin \alpha = 0$

or, $2866.67 \cos \alpha - 533.33 \sin \alpha = 0$

or, $\alpha = 53.34^\circ$

Solving,

$R_B = 533.33N$

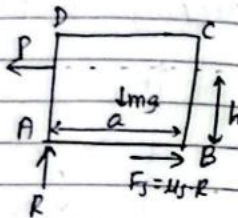
$R_A = 1066.67N$

$F_B = 133.3325N$

Condition of Tipping and Sliding of a block
 here tipping refers to the turning of block ABCD due to the formation of couple. The condition for no tipping is $h = \frac{a}{2\mu_s}$ or $h < \frac{a}{2\mu_s}$

i.e. $h \leq \frac{a}{2\mu_s}$

$h \rightarrow$ point of application of force P above the base of block AB



Proof:

At extreme point of tipping i.e. at just tipping condition, the block will be in equilibrium and reaction R will be at point A.

So, at equilibrium

(i) $\sum M_A = 0$ (2+ve)

or, $mg \times \frac{a}{2} - P \times h = 0$

or, $P \cdot h = mg \frac{a}{2}$ — (1)

(ii) $\sum F_x = 0$ ($\rightarrow$ +ve)

$\mu_s R - P = 0$

or, $P = \mu_s R$ — (2)

(iii) $\sum F_y = 0$ ($\uparrow$ +ve)

$R - mg = 0$

or, $R = mg$ — (3)

Combining (1), (2) & (3)

$h(\mu_s mg) = mg \frac{a}{2}$

or, $h = \frac{a}{2\mu_s}$

So, for no tipping of block (just sliding)

$h \leq \frac{a}{2\mu_s}$

Impending Motion:

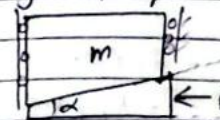
When the force applied on a body becomes equal to maximum frictional force acting on it the body just comes into motion. This stage of motion is called impending motion.

Limiting Friction

Max^m value of frictional force which acts on the body when it just starts to slide over another body is called limiting friction.

Examples of engineering usage of friction

(i) Wedges: Simple machine used to raise heavy loads.



Friction prevents wedge from sliding out

(ii) Friction in Nut-bolt system

(iii) Friction in mechanism of vehicle engine

@8

(iv) Wheel friction or Rolling resistance

$\hookrightarrow$ Significant rolling resistance b/w rubber tyre and tar road.

$\hookrightarrow$ Large rolling resistance due to wet field (Tractor)

⑥

Solution,
Reaction
Calculation

$$(\Sigma M)_{\text{left of hinge}} = 0 (+ve)$$

$$\text{or } 10 \cos 30^\circ \times 3 + R_{Ax} \times 6 = 0$$

$$\text{or } R_{Ax} = -4.33 \text{ kN}$$

$$[R_{Ax} = 4.33 \text{ kN} (\leftarrow)]$$

$$\Sigma F_x = 0 (\rightarrow +ve)$$

$$\text{or } 10 \cos 30^\circ - 4.33 - R_{Fx} = 0$$

$$\text{or } [R_{Fx} = 4.33 \text{ kN} (\leftarrow)]$$

$$\Sigma M_C = 0 (\uparrow +ve)$$

$$\text{or } 4.33 \times 6 - 8.66 \times 3 + 20 \times 2 + 20 + 30 \times 6 - R_{Fy} \times 5 = 0$$

$$\text{or } [R_{Fy} = 48 \text{ kN} (\uparrow)]$$

$$\Sigma F_y = 0 (\uparrow +ve)$$

$$\text{or } R_{Ay} - 10 \sin(30^\circ) - 20 - 30 + 48 = 0$$

$$\text{or } [R_{Ay} = 7 \text{ kN} (\uparrow)]$$

$$\Sigma M_A = 0 (\uparrow +ve)$$

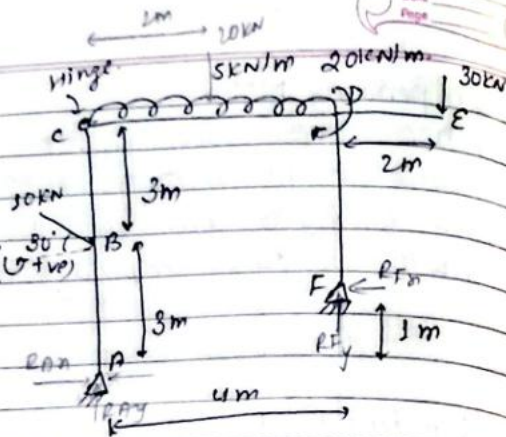
$$\text{or } 10 \cos 30^\circ \times 3 + 20 \times 2 + 20 + 30 \times 6 - 4.33 \times 1 - R_{Fy} \times 4 = 0$$

$$\text{or } [R_{Fy} = 65.41 \text{ kN} (\uparrow)]$$

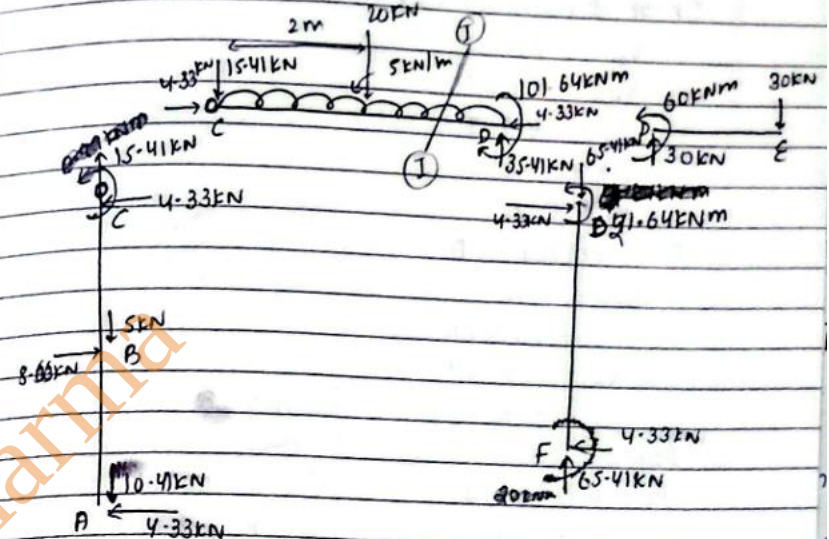
$$\Sigma F_y = 0 (\uparrow +ve)$$

$$\text{or } R_{Ay} - 10 \sin(30^\circ) - 20 - 30 + 65.41 = 0$$

$$\text{or } R_{Ay} = 10.41 \text{ kN}$$



$$[R_{Ay} = 10.41 \text{ kN} (\downarrow)]$$



Axial Force Calculation

For members ABC

$$F_A = 10.41 \text{ kN}$$

$$(F_B)_L = 10.41 \text{ kN}$$

$$F_B = 15.41 \text{ kN}$$

$$(F_C)_L = 15.41 \text{ kN}$$

$$F_C = 0 \text{ kN}$$

For span CD

$$(F_C)_L = 0 \text{ kN}$$

$$F_C = -4.33 \text{ kN}$$

$$(F_D)_L = -4.33 \text{ kN}$$

$$F_D = 0 \text{ kN}$$

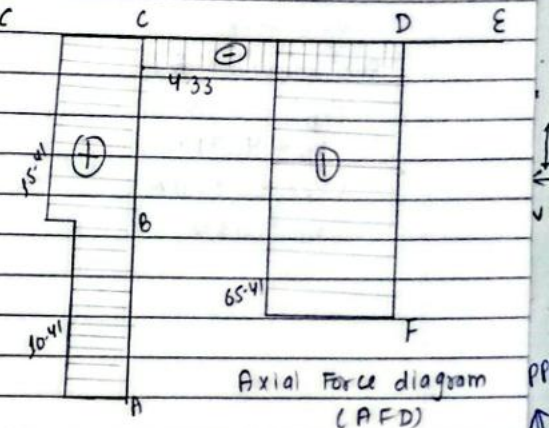
For member DE, $F_{DE} = 0 \text{ kN}$

For members EF,

$$(F_E)_L = 0 \text{ kN}$$

$$F_E = -65.41 \text{ kN}$$

$$(F_F)_L = -65.41 \text{ kN} \quad F_F = 0 \text{ kN}$$



Shear force calculation

For member AB

Shear force just left of A $(V_A)_L = 0 \text{ kN}$

$$V_A = 4.33 \text{ kN}$$

$$(V_B)_L = 4.33 \text{ kN}$$

$$V_B = -4.33 \text{ kN}$$

$$(V_C)_L = -4.33 \text{ kN}$$

$$V_C = 0 \text{ kN}$$

For member CD,

$$(V_C)_L = 0 \text{ kN}$$

$$V_C = -15.41 \text{ kN}$$

$$(V_D)_L = -35.41 \text{ kN}$$

$$V_D = 0 \text{ kN}$$

For member DE,

For member DE,

$$(V_D)_L = 0 \text{ kN}$$

$$V_D = 30 \text{ kN}$$

$$(V_E)_L = 30 \text{ kN}$$

$$V_E = 0 \text{ kN}$$

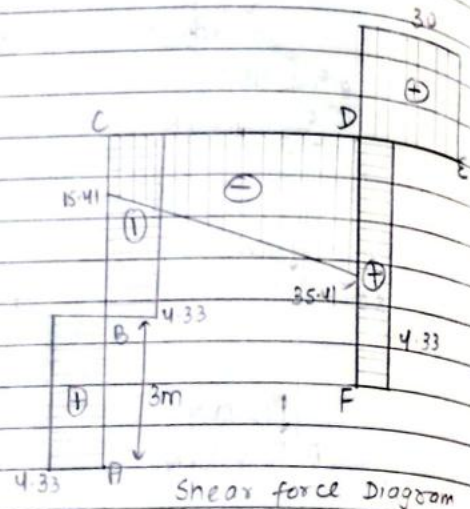
For member EF,

$$(V_D)_L = 0 \text{ kN}$$

$$V_D = 4.33 \text{ kN}$$

$$(V_F)_L = 4.33 \text{ kN}$$

$$V_F = 0 \text{ kN}$$



Bending Moment calculation

For span ABC

Bending moment just left of A $(M_A)_L = 0 \text{ kNm}$

$$M_A = 0 \text{ kNm}$$

$$M_B = 12.99 \text{ kNm}$$

$$(M_C)_L = 0 \text{ kNm}$$

$$M_C = 0 \text{ kNm}$$

For span CDE

$$(M_C)_L = 0 \text{ kNm}$$

$$M_C = 0 \text{ kNm}$$

$$(M_D)_L = -101.64 \text{ kNm}$$

$$M_D = 0 \text{ kNm}$$

For span DE,

$$(M_D)_L = 0 \text{ kNm}$$

$$M_D = -60 \text{ kNm}$$

$$(M_E)_L = 0 \text{ kNm}$$

$$M_E = 0 \text{ kNm}$$

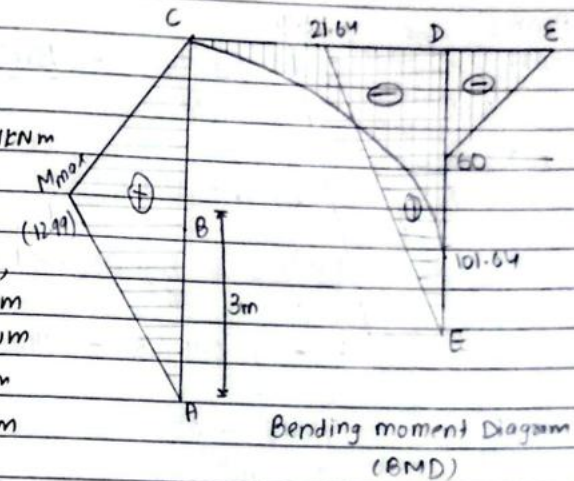
For span EF

$$(M_E)_L = 0 \text{ kNm}$$

$$M_E = -91.64 \text{ kNm}$$

$$(M_F)_L = -20 \text{ kNm}$$

$$M_F = 0 \text{ kNm}$$



$$I_{xc} \rightarrow \bar{y}$$

Parts	y_i	A_i	$y_i A_i$
①	-1.27	-14.13	17.94
②	-1.67	25	-41.75
		$\Sigma A_i = 10.87$	$\Sigma y_i A_i = -23.81$

Frame

ISI: $3 \times \text{total}$
or $3(m+1-j)$
ESI: $3-(3+c)$

Now,
 $\bar{y} = \frac{\Sigma y_i A_i}{\Sigma A_i} = -2.19 \text{ m}$

$$I_{xc} = (I_{xc})_1 - (I_{xc})_2$$

$$= \left[\frac{5^3 \times 10}{36} + 25(-2.19 - (-1.67))^2 \right]$$

$$- \left[0.11(3)^4 + 14.13(-2.19 - (-1.27))^2 \right]$$

$$= 41.48 - 20.87$$

$$I_{xc} = 20.61 \text{ m}^4$$

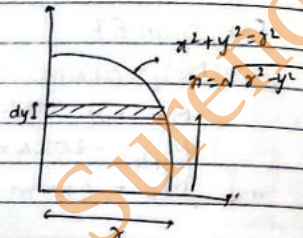
$$A = \int dA$$

$$= \int_0^r \sqrt{r^2 - y^2} dy$$

$$= \left[\frac{y^2}{2} \sqrt{r^2 - y^2} + \frac{r^2}{2} \sin^{-1} \left(\frac{y}{r} \right) \right]_0^r$$

$$= \frac{\pi^2}{2} \times \frac{\pi}{2}$$

$$= \frac{\pi^2}{4}$$



$$\bar{y} = \frac{1}{A} \int y_c dA$$

$$= \frac{4}{\pi r^2} \int_0^r y \sqrt{r^2 - y^2} dy$$

$$= \frac{4}{\pi r^2} \text{ let } y = r \sin \theta$$

$$dy = r \cos \theta d\theta$$

$$y: 0 \rightarrow r$$

$$\theta: 0 \rightarrow \pi/2$$

$$= \frac{4}{\pi r^2} \int_0^{\pi/2} r \sin \theta \cdot r \cos \theta \cdot r \cos \theta d\theta$$

$$= \frac{4}{\pi r^2} \times r^3 \int_0^{\pi/2} \sin \theta \cos^2 \theta d\theta$$

$$= \frac{4r}{\pi} \times \frac{\left[\frac{3}{2} \right] \left[\frac{3}{2} \right]}{2 \times \left[\frac{3}{2} \right]}$$

$$= \frac{4r}{\pi} \times \frac{\frac{1}{2} \sqrt{\pi}}{\frac{2 \times 3 \times \frac{1}{2} \times \sqrt{\pi}}{2 \times 2 \times \sqrt{\pi}}}$$

$$= \frac{4r}{3\pi} \text{ unit}$$

$$\vec{r} = \vec{r}_1 + \vec{r}_2 + \vec{r}_3$$

$$= \vec{ED} \times \vec{F}_{DC} + \vec{ED} \times \vec{F}_{BC} + \vec{ED} \times \vec{F}_{AC}$$

$$= \vec{ED} \times \vec{F}_{DC} + \vec{ED} \times \vec{F}_{BC} + \vec{ED} \times \vec{F}_{AC}$$

$$= (-2\vec{E}) \times \frac{(80\hat{i} - 60\hat{j})}{5} + (-2\vec{E}) \times \frac{(70\hat{i} - 60\hat{j})}{5} + (-2\vec{E}) \times \frac{(3\hat{j}) \times (-120\hat{i})}{5}$$

$$= \frac{160\hat{j} + 160\hat{j} + 240\hat{k} + 160\hat{j} + 160\hat{j} + 240\hat{k} + 240\hat{k} + 240\hat{k}}{5}$$

$$= \frac{320\hat{j} + 480\hat{k}}{5}$$

$$= 64\hat{j} + 96\hat{k}$$

③

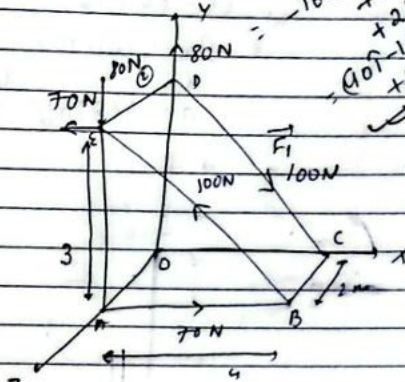
Solution,
let the coordinates of given system be

- O(0,0,0)
- A(0,0,2)
- B(4,0,2)
- C(4,0,0)
- D(0,3,0)
- E(0,3,2)

$$\vec{DC} = 4\hat{i} - 3\hat{j}$$

$$\vec{AE} = 3\hat{j}$$

$$\vec{ED} = -2\hat{k}$$

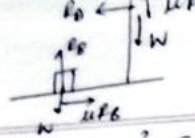


$$\vec{F}_{DC} = 100(\vec{DC})$$

$$= 100(4\hat{i} - 3\hat{j})$$

$$= 400\hat{i} - 300\hat{j}$$

Applying
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$$\theta = 30^\circ$$

$$r = 9000 \text{ m}$$

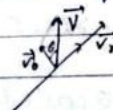
$$\dot{\theta} = 0.02 \text{ rad/s}$$

$$\ddot{\theta} = 0.002 \text{ rad/s}^2$$

$$\vec{v} = \dot{r} \hat{e}_r + r \dot{\theta} \hat{e}_\theta$$

$$\vec{a} = (\ddot{r} - r \dot{\theta}^2) \hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta}) \hat{e}_\theta$$

$$\frac{v_\theta}{v} = \omega r$$



$$v_\theta = r \dot{\theta} = 9000(0.02) = 180 \text{ m/s}$$

$$v = 207.84 \text{ m/s}$$

$$\frac{v_\theta}{v} = \sin 30^\circ$$

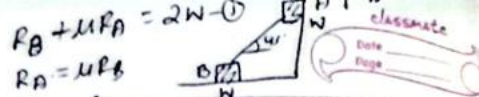
$$\Rightarrow v_\theta = \dot{r} = v \sin 30^\circ = 103.93 \text{ m/s}$$

$$(\ddot{r} - r \dot{\theta}^2) \hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta}) \hat{e}_\theta$$

$$\frac{a_\theta}{a} = \cos(70^\circ)$$

$$a_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta}$$

$$a = 64.78 \text{ m/s}^2 = 22.16 \text{ m/s}^2$$



$$R_\theta + \mu R_A = 2W \quad (1)$$

$$R_A = \mu R_B$$

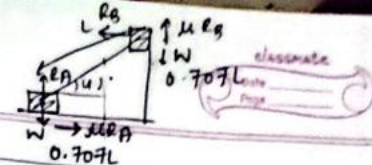
$$R_A = R_B$$

$$\mu R_A + \mu R_B = 2W$$

$$\mu + \mu = 2W$$

$$N = -0.707T$$

$$W + 0.707T = 0$$



$$\Sigma M_A = 0 \quad (\uparrow +ve)$$

$$\text{or, } W \times 0.707L - \mu R_B \times 0.707L - R_B \times 0.707L = 0$$

$$\text{or, } W - \mu R_B - R_B = 0$$

$$W = \mu R_B + R_B \quad (1)$$

$$\Sigma F_y = 0$$

$$R_A + \mu R_B = 2W$$

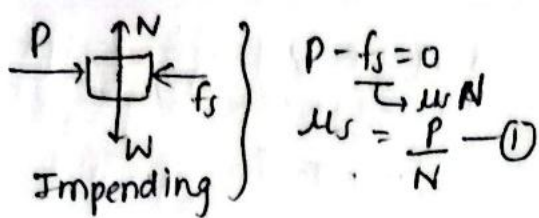
$$\text{or, } R_A + \mu R_B = \mu R_B + R_B$$

$$R_A = R_B$$

$$\Sigma F_x = 0$$

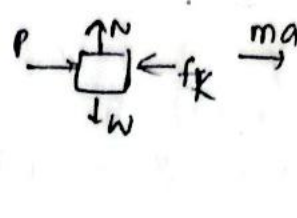
$$\mu R_A = R_B$$

$$\mu = 1$$



$$P - f_s = 0$$

$$\mu_s = \frac{f_s}{N} = \frac{P}{N} \quad \text{--- (1)}$$



$$P - f_k = ma$$

$$\Rightarrow \mu_k = \frac{P - ma}{N}$$

$$a > 0, \mu_s > \mu_k$$

(Q) Find smallest magnitude of P that will cause impending motion (tipping or slipping) of drum.

$$\rightarrow W = 100 \text{ lb}, \mu_s = 0.5, a = 3 \text{ ft}, b = 4 \text{ ft}$$

Sliding case

$$100 + P \sin 37^\circ = N$$

$$\Rightarrow N = 100 + 0.6P \quad \text{--- (1)}$$

Impending

$$P \cos 37^\circ = F \quad \Sigma F_x$$

$$0.8P = \mu_s(N)$$

$$\text{or, } 0.8P = 0.5(100 + 0.6P)$$

$$\text{or, } 0.8P = 50 + 0.3P$$

$$\Rightarrow P = 100 \text{ lb}$$

$$N = 160 \text{ lb}$$

Now,

Let drum slips; $x \leq 1.5 \text{ ft}$.

$$\Sigma M_o = 0 \quad (\curvearrowright +ve)$$

$$-P \sin(37^\circ) \times 1.5 - N \times x + 100 \times 4 + P \cos 37^\circ \times 4 = 0$$

$$\text{or, } x = 1.4 \text{ ft} \leq 1.5 \text{ so slips.}$$

Tipping case

Let drum tips.

$$x = 1.5 \text{ ft (from o)} \quad \{ \text{Min}^m \}$$

$$\star \text{ (i) } \Sigma M_{\text{edge}} = 0 \quad (\curvearrowright +ve)$$

$$P \cos(36.86^\circ) \times 4 - P \sin(36.86^\circ) \times 3 - 100 \times 1.5 = 0$$

$$\Rightarrow P = 107.08 \text{ N} \rightarrow \text{Min}^m \text{ force for tipping.}$$

$$\text{(ii) } \Sigma F_y = 0$$

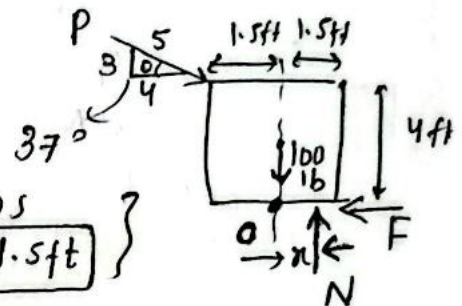
$$N - P \sin 36.87 - 100 = 0$$

$$\Rightarrow N = 164.285 \text{ lb}$$

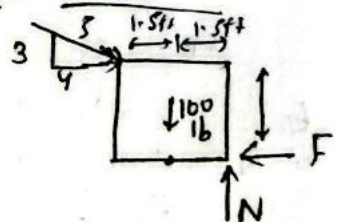
$$F_{\text{lim}} = \mu_s \times N = 0.5 \times 164.285 = 82.142 \text{ N}$$

$$\text{(iii) } \Sigma F_x = 0 \Rightarrow P \cos(36.87^\circ) - F = 0$$

$$\Rightarrow F = 85.713 \text{ lb} > F_{\text{lim}} \rightarrow \text{Thus, drum has slipped.}$$



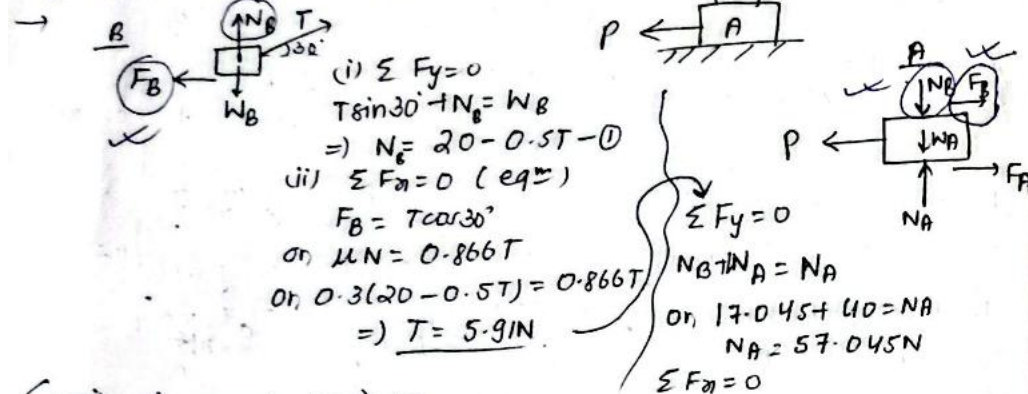
★ Already known that drum slips so, no need to do tipping case



tipping.

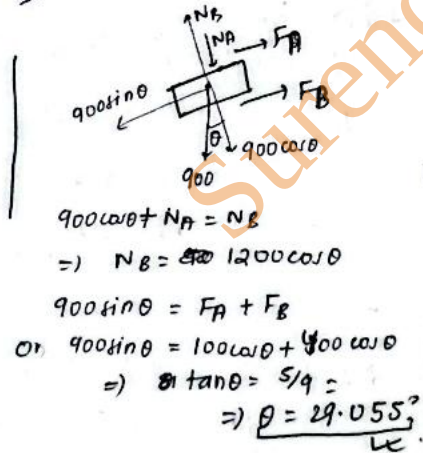
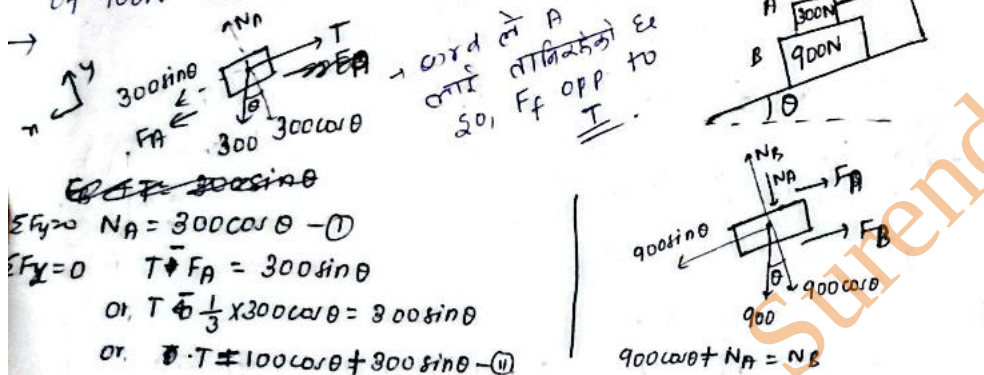
limiting frictional force (drum not moved yet)

3 Two blocks A and B of 40N and 20N resp are in eq^m position as shown. Calculate the force P required to move block A. Take $\mu = 3$ for all surface.

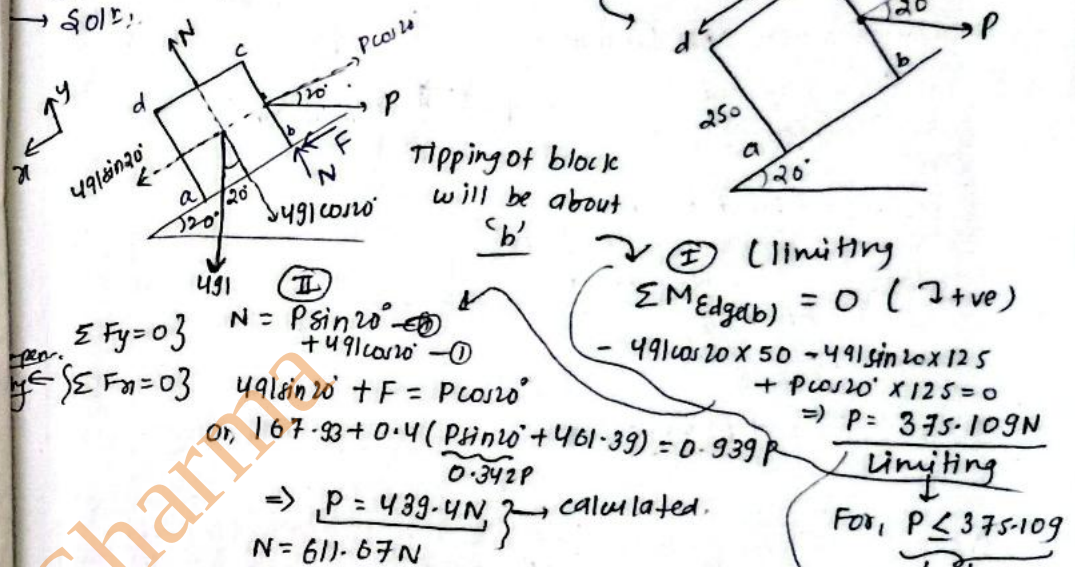


जहाँ जहाँ touch होगा वहाँ वहाँ reaction आएगा

3 What should be value of θ which will make the motion of 900N block down the plane impen. $\mu = 1/3$ for all contact surface.



The homogeneous block has a mass of 50kg and the coefficient of friction b/w the block and plane is 0.4. If the force 'P' is increased gradually until motion ensues, will the block slide or tip and for what value of P.



$P_{\text{calculated}} > P_{\text{limiting}}$
 Block ~~slides~~ Tipples

(or) After P_{lim} $P = 375.109N$

(ii) $\sum F_y = 0$ $N = P \sin 20^\circ + 491 \cos 20^\circ$
 $= 589.68N$

$F_{\text{lim}} = \mu N = 235.87N$

Now,

(iii) $\sum F_x = 0$

$F = P \cos 20^\circ - 491 \sin 20^\circ = 184.56N$

$F_{\text{lim}} > F$ So, there will not be sliding
 Block tipples

5 For Ladder

what value of P eq^m condⁿ is maintained?

$\Rightarrow \mu$

→ Frictional force → opp. to motion

vertical $\Rightarrow$ 2 unknown

Hor $\Rightarrow$ 3 unknown.

So, find

(1) $\Sigma M_A = 0$ (Find Normal Rxn)

↳ frictional force at B

(2) $\Sigma F_y = 0$ { Find N_A }
 $N_B \uparrow, f_B \leftarrow$
 $f_A \leftarrow$

(3) $\Sigma F_x = 0$ (Find P) $\leftarrow$

Or, $\Sigma M_B = 0$

Q Determine min^m angle θ at which a uniform ladder can be placed against a wall without slippage under its own wt. coeff of friction ' μ ' for all surfaces

Applying eq^m eqⁿ.

(1) $\Sigma M_A = 0$ (2 + ve)

find N_B & f_B in terms of θ

$$(W \cdot \frac{L}{2} \cos \theta) - (\mu \cdot N_B \cdot L \cos \theta) - (N_B \cdot L \sin \theta) = 0 \quad (1)$$

(2) $\Sigma F_y = 0$

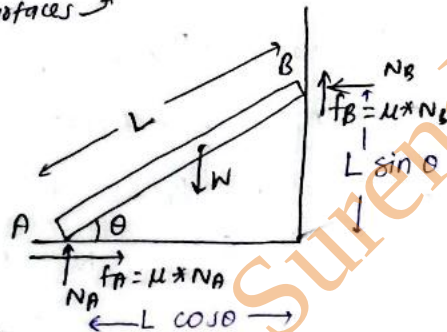
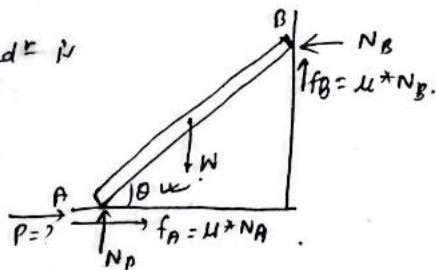
↳ find N_A (in terms of θ)

$$N_A + \mu N_B - W = 0 \quad (2)$$

(3) $\Sigma F_x = 0$

$$\mu N_A - N_B = 0 \quad (3)$$

↳ find θ



If $\mu = 0.2$ Given,

From (2) & (3) $\rightarrow N_A = \frac{N_B}{\mu} = 5N_B \leftarrow$

(2) $\rightarrow W = N_A + \mu N_B = 5.2N_B \leftarrow$

(1) $(5.2N_B) \cdot \frac{L}{2} \cos \theta - (0.2N_B \cdot L \cos \theta) - (N_B \cdot L \sin \theta) = 0$

Or, $5.2 \cos \theta - 0.2 \cos \theta = 0.2 \sin \theta$

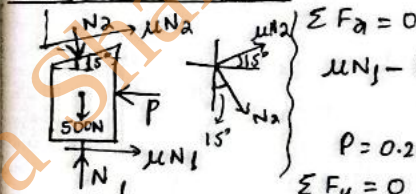
$\Rightarrow \theta = \tan^{-1} \left(\frac{2.4 N_B}{N_B} \right) = 67.38^\circ$

Wedge Friction

Q. Block A weighing 1000N is to be raised by means of a 15° wedge B weighing 500N. Assuming coefficient of friction b/w all contact surfaces to be 0.2, find minimum horizontal force P should be applied to raise the block.

→ Solⁿ:
 $\Rightarrow$ If wt of wedge not given consider weightless.
 $\Rightarrow$ Draw FBD where ' P ' is applied first

FBD of wedge B



$\Sigma F_x = 0$

$$\mu N_1 - P + \mu N_2 \cos(15^\circ) + N_2 \sin(15^\circ) = 0$$

$$P = 0.2N_1 + 0.452N_2 \quad (i)$$

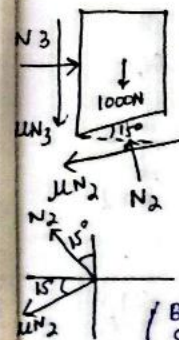
$\Sigma F_y = 0 \rightarrow 0.2N_1 + 0.452N_2 - P = 0 \quad (ii)$

$$500 - N_1 + N_2 \cos(15^\circ) - \mu N_2 \sin(15^\circ) = 0$$

Or, $N_1 = 500 + 0.914N_2 \quad (iii)$

$$N_1 - 0.914N_2 = 500$$

FBD of Block A



$\Sigma F_x = 0$

$$N_3 - \mu N_2 \cos(15^\circ) - N_2 \sin(15^\circ) = 0$$

$$-0.742N_2 + N_3 = 0 \quad (iii)$$

$\Sigma F_y = 0$

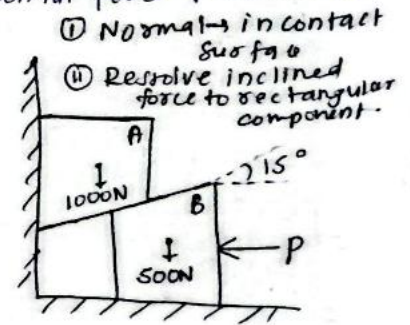
$$1000 - \mu N_3 + \mu N_2 \sin(15^\circ) - N_2 \cos(15^\circ) = 0$$

Or, $0.914N_2 + 0.2N_3 = 1000 \quad (iv)$

From (i), (ii), (iii) & (iv)

$$N_1 = 1360.32N, N_2 = 44.26N,$$

$$N_3 = 698.42N, P = 697.52N$$



Three unknowns
 N_1, N_2 & P

Dynamics

t	x	v	a

Any two values given in order to find value of constant of given function.

1. The accelⁿ of particle is given by $a = -kx$, at the initial position velocity is 2 m/s. Determine the position as function of time when position is 4m, velocity is 0 m/s.

→ Soln:

$$a = -kx \quad \{ a = f(x) \} \Rightarrow v \frac{dv}{dx} = f(x) \quad \text{given position \& velocity}$$

$$\text{or } v \frac{dv}{dx} = -kx$$

$$\text{or } \int_2^0 v dv = \int_0^4 -kx dx \quad \begin{cases} x=0 \rightarrow v=2 \text{ m/s} \\ x=4 \rightarrow v=0 \text{ m/s} \end{cases}$$

$$\text{or } \left[\frac{v^2}{2} \right]_2^0 = -k \left[\frac{x^2}{2} \right]_0^4$$

$$\text{or } -\frac{2^2}{2} = -k \frac{4^2}{2} \Rightarrow k = \frac{1}{4}$$

$$\therefore a = -\frac{x}{4}$$

$$\int_2^0 v dv = -\int_0^4 \frac{x}{4} dx$$

$$\Rightarrow \frac{v^2}{2} - \frac{2^2}{2} = -\frac{1}{4} \left[\frac{x^2}{2} - 0 \right] \quad \text{would be easier.}$$

$$\text{or } \frac{v^2 - 4}{2} = -\frac{x^2}{8}$$

$$\Rightarrow 4v^2 = 16 - x^2$$

$$\text{or } v^2 = \frac{16 - x^2}{4}$$

$$\Rightarrow v = \frac{1}{2} \sqrt{16 - x^2}$$

$$\text{or } \frac{dx}{dt} = \frac{1}{2} \sqrt{16 - x^2}$$

$$\int_0^x \frac{dx}{\sqrt{16 - x^2}} = \int_0^t \frac{1}{2} dt \Rightarrow 2 \left[\sin^{-1} \frac{x}{4} \right]_0^x = \frac{1}{2} [t]_0^t$$

$$\sin^{-1} \frac{x}{4} \Big|_0^x = \frac{1}{4} [t]_0^t$$

$$\text{for } t_0 = 0, x_0 = 0 \Rightarrow x = 4 \sin \left(\frac{t}{4} \right)$$

Eqⁿ req^d further?

Yes $\Rightarrow$ Then put limit both in variables

Here,

$$\int_{v_0}^v v dv = \int_{x_0}^x -kx dx$$

would be easier.

Just put v_0 & x_0 to find eqⁿ.

$v = f(x)$ & put other v & x to find constant.

$$x = f(t) \quad \text{find constant.}$$

2. The acceleration of a particle is defined by relation $a = -k\sqrt{v}$ where k is constant knowing that $x=0$ & $v=8 \text{ m/s}$ at $t=0$ and that $v=36 \text{ m/s}$ when $x=18 \text{ m}$. Determine
(a) Velocity of particle when $x=20 \text{ m}$
(b) Time req^d for particle to come to rest.

→ Soln:-

$$a = -k\sqrt{v} \quad \text{--- (I)}$$

$$\text{or } v \frac{dv}{dx} = -k\sqrt{v}$$

$$\text{or } \int_8^v \sqrt{v} dv = -\int_0^x k dx$$

$$\text{or } \frac{2}{3} [(v)^{3/2}]_8^v = -k [x]_0^x$$

$$\text{or } [(v)^{3/2} - 72] = -\frac{3k}{2} x \quad \text{--- (II)}$$

$$\text{When } x=18 \text{ m, } v=36 \text{ m/s}$$

$$\text{So, } k = \frac{1}{19}$$

$$\text{Thus } [(v)^{3/2} - 72] = -\frac{57}{2} x \quad \text{--- (III)}$$

$$\text{Thus } [(v)^{3/2} - 72] = -\frac{57}{2} x \quad \text{--- (III)}$$

$$\text{(I) For } x=20 \text{ m, } v=29.35 \text{ m/s}$$

$$\text{(II) From (II) For } v=0, \frac{v}{2} = [72 - 28.5x]$$

$$\frac{dv}{dt} \text{ For } v=0 \Rightarrow x = 25.579 \text{ m}$$

From (II).

$$a = -19\sqrt{v}$$

$$\int_8^v \frac{dv}{\sqrt{v}} = -19 \int_0^t dt$$

$$\text{or } 2[(v)^{1/2}]_8^v = -19[t]_0^t$$

$$\text{or } 2[\sqrt{v} - 2.828] = -19t$$

$$\text{for } v=0, t = 0.947 \text{ sec.}$$

Eqⁿ of v used further so, don't put limit directly

3. The acceleration of a particle is given by relation $a = v^3$. It is known that at time $t = 0$, position is -2m & velocity is 1m/s . Find displacement, position, velocity & acceleration at instant of $1/4\text{sec}$.

→ Soln:-

$$\text{or, } \int_1^v \frac{dv}{v^3} = \int_0^t dt \quad \left\{ \begin{array}{l} \text{At some time} \\ \text{So, if } t \text{ then every eq}^n \end{array} \right.$$

$$\text{or, } \left[\frac{v^{-2}}{-2} \right]_1^v = [t]_0^t$$

$$\text{or, } \left[\frac{1}{-2v^2} + \frac{1}{1} \right] = t$$

$$\text{or, } \frac{1}{-2v^2} = t - 1$$

$$\text{or, } v = \sqrt{\frac{1}{-2(t-1)}}$$

$$\text{or, } v = \sqrt{\frac{1}{2-2t}} \quad \text{--- (1)}$$

$$\frac{dx}{dt} = \frac{1}{\sqrt{2-2t}}$$

$$\text{or, } \int_{-2}^x -2dx = \int_0^t \frac{1}{\sqrt{2-2t}} dt$$

$$\text{or, } -2[x]_{-2}^x = [2\sqrt{2-2t}]_0^t$$

$$\text{or, } -2[x+2] = 2\sqrt{2-2t}$$

$$\text{or, } [x = -2 - \sqrt{2-2t}] \quad \text{--- (1)}$$

1) Position: At $t = 1/4\text{sec} \Rightarrow x = -3.225\text{m}$

2) Velocity: At $t = 1/4\text{sec} \Rightarrow v = 0.816\text{m/s}$

3) Acceleration At $t = 1/4\text{sec} \Rightarrow v = 0.816\text{m/s} \Rightarrow a = 0.54\text{m/s}^2$

4) Displacement: At $t = 1/4\text{sec} \Rightarrow \frac{x_{1/4} - x_0}{\Delta t} = \frac{-3.225 - (-2)}{1/4} = -1.225\text{m/s}$

Motion of several particles: Several particles move independently

(a) Relative position of particle B relative to A

$$x_{B/A} = x_B - x_A \quad \text{--- (1)}$$

$$x_B = x_{B/A} + x_A$$

+ve sign of B/A $\Rightarrow$ B is at right of A

-ve sign of B/A $\Rightarrow$ B is at left of A

$$v_B = v_A + v_{B/A} \quad \text{--- (2)}$$

$v_{B/A} \rightarrow$ Rel. velocity of B w.r.t A

+ve $\rightarrow$ B is observed from A to move in +ve dir

-ve $\rightarrow$ B is observed from A to move in -ve dir

$$a_B = a_A + a_{B/A} \quad \text{--- (3)}$$

(b) Dependent motion: position of particle will depend upon position of another or several other particles.

Eg:

$\rightarrow$ Length of rope ACDEFG is constant.

$\rightarrow$ position of block 'B' depends on position of block 'A'

$$HC = EB = \text{const.}$$

Length of ACDEFG = const.

$$\Rightarrow AC + CD + DE + EF + FG = \text{const.}$$

$$\text{or, } AC + DE + FG = \text{const.}$$

$$\text{or, } (x_A - HC) + (x_B - HC - FB) + (x_B - FB) = C$$

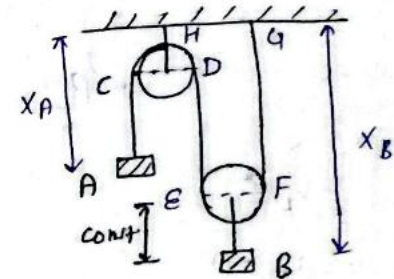
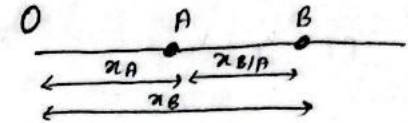
$$\text{or, } x_A + 2x_B = \text{const} \quad \text{--- (1)}$$

$$v_A + 2v_B = 0 \quad \text{--- (2)}$$

$$v_A = -2v_B$$

$$a_A + 2a_B = 0 \quad \text{--- (3)}$$

* If Block 'A' moves downward then 'B' moves upward & vice-versa.

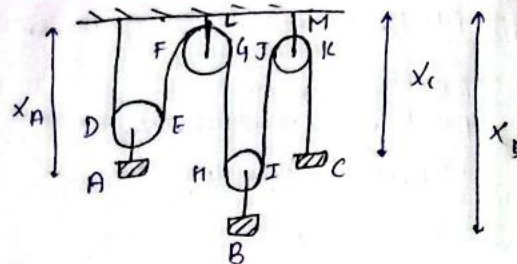


eg ②

$$2x_A + 2x_B + x_C = C \quad \text{--- (I)}$$

$$2v_A + 2v_B + v_C = 0 \quad \text{--- (II)}$$

$$2a_A + 2a_B + a_C = 0 \quad \text{--- (III)}$$



$v_0 = 0$

Q. Block 'C' starts from rest and move downward with constant acceleration, knowing that after 12 sec, velocity of

block A is 456 m/s, Determine

(a) Accⁿ of A, B, C

(b) velocity of change in pos. of block B after 8 sec.

$$2x_A + 4x_B + 2x_C = C$$

Two ropes ★

Rope (I)

$$2x_A + 3x_B = C \quad \text{--- (I)}$$

Rope (II)

$$x_B + 2x_C = C \quad \text{--- (II)}$$

$$2v_A + 3v_B = 0$$

$$\Rightarrow v_B = -\frac{2}{3}v_A \quad \text{--- (III)}$$

$$a_B = -\frac{2}{3}a_A \quad \text{--- (IV)}$$

Q From question acceleration is const. (C & A both)

$$\therefore v_A = (v_A)_0 + a_A t$$

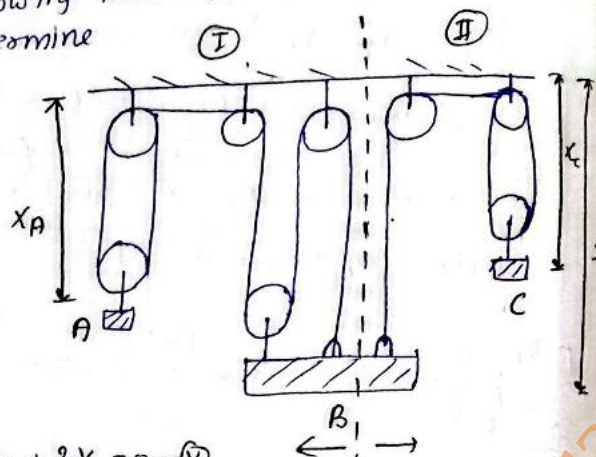
$$a_A = \frac{v_A - (v_A)_0}{t} = \frac{456 - 0}{12} = 38 \text{ m/s}^2 (\downarrow)$$

$$\text{Now, } a_B = -\frac{2}{3} \times 38 = -25.33 = 25.33 \text{ m/s}^2 (\uparrow)$$

$$a_C = -\frac{a_B}{2} = 12.67 \text{ m/s}^2 (\downarrow)$$

$$(b) v_B = (v_B)_0 + a_B t \quad \left\{ \because \text{Block 'B' starts from rest} \right\}$$

$$v_B = 25.33 \times 8 = 202.64 \text{ m/s} (\uparrow) \quad (v_B)_0 = 0 \text{ m/s}$$



Normal & Tangential component

$$\vec{v} = v \hat{e}_t + 0 \hat{e}_n \quad \text{--- (1)}$$

$$\vec{a} = \frac{dv}{dt} \hat{e}_t + \frac{v^2}{\rho} \hat{e}_n = a_t \hat{e}_t + a_n \hat{e}_n \quad \text{--- (11)}$$

$$\rho = \frac{[1 + (\frac{dy}{dx})^2]^{3/2}}{\frac{d^2y}{dx^2}}$$

Q A car is travelling down a slopping road represented by the curve $x^2 = 300y$, where x & y are in m, when car is 12 m above the ground velocity is 60 km/hr & is decelerating at rate of 2 m/s². Determine the total acceleration at this instant.

→ Solⁿ.

$$x^2 = 300y \quad \text{--- (1)}$$

↳ Curved path

(Two components → normal & tangential)

$$v = 60 \text{ km/hr} = 16.67 \text{ m/s}$$

$$y = 12 \text{ m} \quad \left. \begin{array}{l} \Rightarrow x = 60 \text{ m} \end{array} \right\} \text{Position}$$

$$x^2 = 300y$$

Diff. w.r.t. x

$$2x = 300 \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{x}{150} \quad \text{--- (2)}$$

$$\rightarrow 0.4$$

$$\frac{d^2y}{dx^2} = \frac{1}{150} \quad \text{--- (3)}$$

$$\rightarrow 0.0067$$

$$\rho = \frac{[1 + (\frac{dy}{dx})^2]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$\rho = 187.31 \text{ m/s}$$

$$a_t = -2 \text{ m/s}^2 \text{ (decelerating)}$$

$$a_n = \frac{v^2}{\rho} = \frac{16.67^2}{187.31} = 1.484 \text{ m/s}^2$$

$$a = \sqrt{a_t^2 + a_n^2} = 2.49 \text{ m/s}^2$$

$$\theta = \tan^{-1}(\frac{a_n}{a_t}) = -36.58^\circ \text{ (with } \hat{e}_t)$$

Q The rectangular components of acceleration for a particle are $a_x = 3t$ and $a_y = (30 - 10t)$, where 'a' is in m/s². If particle starts from rest at origin. Find radius of curvature of path at instant of 2 sec.

→ Solⁿ.

$$\text{At } t=0, \quad x=0, y=0 \quad u_x=0, u_y=0$$

$$a_n = \frac{v^2}{\rho} \rightarrow v = v_R \quad \text{resultant velocity}$$

$$a_t \perp a_n$$

$$v_x = u_x + a_x t \quad | \quad t=2$$

$$v_x = 0 + (3 \times 2) \times 2 = 12 \text{ m/s}$$

$$v_y = u_y + a_y t$$

$$= 0 + (30 - 10 \times 2) \times 2$$

$$= 20 \text{ m/s}$$

$$v_R = v = \sqrt{v_x^2 + v_y^2} = 23.32 \text{ m/s} \quad \text{At } t=2 \text{ sec}$$

$$\text{And } a_R = a = \sqrt{a_x^2 + a_y^2} = \sqrt{6^2 + 10^2} = 11.66 \text{ m/s}^2 \quad \text{At } t=2 \text{ sec}$$

$$\text{After } t=2 \text{ sec } \left\{ a_t = \frac{dv}{dt} \right\} = \frac{23.32}{2} = 11.66 \text{ m/s}^2 \quad \text{Initially rest so, } u_0=0$$

$$a = \sqrt{a_t^2 + a_n^2} \Rightarrow a_n = 0.2107 \text{ m/s}^2$$

$$a_n = \frac{v^2}{\rho}$$

$$\Rightarrow \rho = 2580.866 \text{ m}$$

→ At $t=2 \text{ sec}$.

$$a_x = 3t$$

$$\int_0^v dv = \int_0^2 3t dt$$

$$\text{or, } v_x = 3 \left[\frac{t^2}{2} \right]_0^2$$

$$v_x = 6 \text{ m/s}$$

$$a_n = 30 - 10t$$

$$\int_0^v dv = \int_0^2 (30 - 10t) dt$$

$$v_y = [30t - 5t^2]_0^2$$

$$v_y = 40 \text{ m/s}$$

$$a_x = 6 \text{ m/s}^2 \quad | \quad a_y = 10 \text{ m/s}^2$$

Starts from rest at origin

$$a_n = 6 \text{ m/s}^2$$

$$a_y = 10 \text{ m/s}^2$$

$$V_R = V = \sqrt{V_x^2 + V_y^2} = 40.45 \text{ m/s}$$

$$a_t = \frac{dV}{dt} = \frac{40.45}{2} = 20.225 \text{ m/s}^2$$

$$\text{Direction } (\alpha) = \tan^{-1}\left(\frac{V_y}{V_x}\right) = \tan^{-1}\left(\frac{40}{6}\right) = 81.47^\circ$$

$$a_t \perp a_n, \quad a_t \text{ along } V$$

Then,

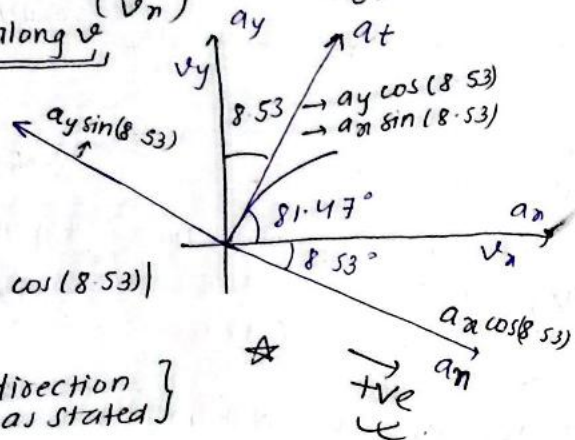
$$a_t = a_y \cos(8.53) + a_n \sin(8.53)$$

$$= 8.80 \text{ m/s}^2$$

$$a_n = |a_y \sin(8.53) - a_n \cos(8.53)|$$

$$= |1 - 4.45|$$

$$= 4.45 \text{ m/s}^2 \quad \left\{ \begin{array}{l} \text{In direction} \\ \text{as stated} \end{array} \right\}$$



$$\text{So, } a_n = \frac{V^2}{r} \Rightarrow r = \frac{V^2}{a_n} = \frac{40.45^2}{4.45} = 367.686 \text{ m}$$

A nozzle discharges a stream of water in the direction shown in figure with an initial velocity of 25 m/s.

Determine radius of curvature of stream

(a) as it leaves the nozzle

(b) as the max^m ht of stream

→ Solⁿ:

Projectile motion.

$$y = x \tan \theta - \frac{g x^2 \sec^2 \theta}{2 V_0^2}$$

$\theta \rightarrow$ Angle of projection (with horizontal)

$V_0 \rightarrow$ velocity (initial)

Initially $\theta = \tan^{-1}\left(\frac{3}{4}\right) = 36.86^\circ$

So, $y = 0.75x - 0.01226x^2$ — (i)

$$\frac{dy}{dx} = 0.75 - 0.024525x$$
 — (ii)
$$\frac{d^2y}{dx^2} = -0.024525$$
 — (iii)

(a) Initially, $x=0, y=0$

So, $\frac{dy}{dx} = 0.75 \quad \left| \quad \frac{d^2y}{dx^2} = -0.024525 \right.$

$$\text{So, } r = \left| \frac{[1 + \left(\frac{dy}{dx}\right)^2]^{3/2}}{\frac{d^2y}{dx^2}} \right| = 79.64 \text{ m}$$

(b) At max^m ht.

$\downarrow$ ~~$\frac{dy}{dx} = 0$~~ $V_y^2 = u_y^2 - 2gy$

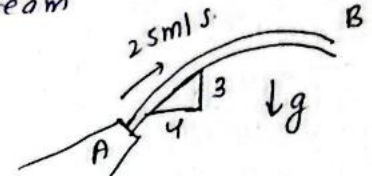
$\frac{dy}{dx} = 0$ $\downarrow$ 0 m/s $\downarrow$ 20 m/s

$y = 20.387 \text{ m}$

So, From (ii) $\frac{dy}{dx} = 0$

$x = 30.58 \text{ m}$ At 30.58 m, $\frac{d^2y}{dx^2} = -0.024525$

So, $r = \left| \frac{[1 + \left(\frac{dy}{dx}\right)^2]^{3/2}}{\frac{d^2y}{dx^2}} \right| = 40.77 \text{ m}$



OP Using Tangential and Normal component

$$v_0 = 25 \text{ m/s}$$

$$\theta = 36.87^\circ$$

$$(v_0)_x = 20 \text{ m/s}$$

$$(v_0)_y = 15 \text{ m/s}$$

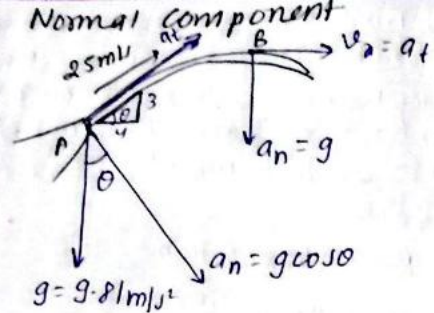
Case (i): Initially

$$\rho = \frac{v^2}{a_n} \rightarrow v_t = \underline{v_0}$$

$$= \frac{(25)^2}{9.8 \cos \theta} = \underline{79.64 \text{ m}}$$

Case (ii): At max height

$$\rho = \frac{v_x^2}{a_n} = \frac{20^2}{9.81} = \underline{40.77 \text{ m}}$$



$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta$$

$$\vec{a} = -13.11\hat{e}_r + 26.18\hat{e}_\theta$$

$$|\vec{a}| = 29.28 \text{ m/s}^2$$

$$\phi = \tan^{-1}\left(\frac{a_\theta}{a_r}\right) = -63.39^\circ \text{ (with radial component)}$$

Q. The relation for r & θ for motion of a particle is given by $r = \theta^2$ & $\theta = t^2$ where r is in meter, θ in radian, t in sec. Find velocity and acceleration when $\theta = 0.2$ radian.

→ Solⁿ

$$\begin{aligned} r &= \theta^2 = t^4 = 0.0399 \text{ m} & \theta &= t^2 = 0.2 \text{ rad} \\ \text{At } t &= 0.447 \text{ sec} & \dot{\theta} &= 0.4 \text{ rad/sec} & \ddot{\theta} &= 2 \text{ rad/sec}^2 \\ \dot{r} &= 0.4t^3 = 0.357 \text{ m/s} & \ddot{r} &= 1.2t^2 = 2.398 \text{ m/s}^2 \end{aligned}$$

$$\text{Now, } \vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta = (0.357\hat{e}_r + 0.0357\hat{e}_\theta) \text{ m/s}$$

$$|\vec{v}| = 0.3588 \text{ m/s}$$

$$\alpha = \tan^{-1}\left(\frac{v_\theta}{v_r}\right) = 5.71^\circ$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta$$

$$\vec{a} = (2.366\hat{e}_r + 0.718\hat{e}_\theta) \text{ m/s}^2$$

$$|\vec{a}| = 2.47 \text{ m/s}^2$$

$$\beta = \tan^{-1}\left(\frac{a_\theta}{a_r}\right) = 16.88^\circ$$

Given in r & θ

radial & transverse component

$\dot{\theta} \rightarrow$ in radian

Radial And Transverse component:

Q. A particle moves along a curved path defined by $r = 5\theta$ & $\theta = t^2/3$ where ' r ' is in meter and ' t ' in second. Make calculation for velocity and accelⁿ when $\theta = \pi/2$.

→ Solⁿ:

$$\text{For } \theta = \pi/2, t = 2.17 \text{ sec}$$

$$\theta = t^2/3 \quad r = 5\theta = \frac{5t^2}{3}$$

$$\dot{\theta} = 2t/3 \quad \dot{r} = 10t/3$$

$$\ddot{\theta} = 2/3 \quad \ddot{r} = 10/3$$

$$\text{At } t = 2.17 \text{ sec, } \dot{\theta} = 1.447 \text{ rad/sec} \mid \dot{r} = 7.237 \text{ m/sec} \mid r = 7.855 \text{ m}$$

$$\theta = 1.571 \text{ rad}$$

$$\text{Now, } \vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

$$\vec{v} = (7.237\hat{e}_r + 11.366\hat{e}_\theta) \text{ m/s}$$

$$|\vec{v}| = 13.475 \text{ m/s}$$

$$\alpha = \tan^{-1}\left(\frac{v_\theta}{v_r}\right) = 57.51^\circ$$

(with radial component)

Q. A stone is thrown from top of a building upward at an angle of 40° to the horizontal with an initial speed of 30 m/s as shown in figure. Determine

(a) horizontal distance from pt. of projection to the pt. where stone strikes the ground.

(b) Greatest elevation reached by projectile above ground.

(c) velocity when it strikes the ground.

(d) Time of flight.

→ Solⁿ:

$$(a) \quad y = (u_y)_0 t + \frac{1}{2} g t_{\max}^2 \quad (\downarrow +ve)$$

$$\text{on } 30 = -19.28t + 4.905t^2$$

Solving,

$$t = 5.124 \text{ sec} \rightarrow T_{\max} (\text{time of flight})$$

$$(a) \quad x(R) = (u_x)_0 \times T$$

$$R = 117.75 \text{ m}$$

(b) At greatest elevation, $(v_y)_0 = 0$

$$(v_y)^2 = (v_0)^2 - 2g H_{\max} \quad \text{above elevation level (projection)}$$

$$0 = 19.28^2 - 2 \times 9.81 H_{\max}$$

$$\rightarrow H_{\max} = 18.95 \text{ m}$$

$$\text{Above ground: } 18.95 + 30 = 48.95 \text{ m}$$

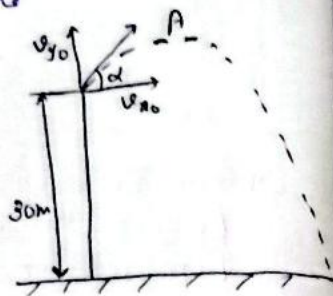
At ground

$$(c) \quad (v_f)_x = (v_0)_x = 22.98 \text{ m/s}$$

$$(v_f)_y = (v_0)_y - gT = -31.87 \text{ m/s} \quad (\uparrow +ve)$$

$$\text{Thur, } v = \sqrt{v_x^2 + v_y^2} = 39.29 \text{ m/s}$$

$$\alpha = \tan^{-1} \left(\frac{v_y}{v_x} \right) = -54.21^\circ$$



Q. A projectile is fired with an initial velocity of 200 m/s to hit a target located 500 m above the level of a gun point & at horizontal distance of 3000 m . Neglect air resistance, determine firing angle.

→ Solⁿ:

Eqs of projectile motion

$$y = x \tan \theta - \frac{g x^2 \sec^2 \theta}{2 v_0^2}$$

$$\theta = \alpha$$

$$y = 500 \text{ m}, x = 3000 \text{ m}, v_0 = 200 \text{ m/s}$$

$$g = 9.81 \text{ m/s}^2$$

$$\sec^2 \alpha = (1 + \tan^2 \alpha)$$

So,

$$500 = 3000 \tan \alpha - \frac{(1103.625)(1 + \tan^2 \alpha)}{200^2}$$

$$\text{or, } 500 \cdot 1103.625 \tan^2 \alpha - 3000 \tan \alpha + 1603.625 = 0$$

Solving,

$$\tan \alpha = 1.987$$

$$|\tan \alpha = 0.73125$$

$$\alpha = 63.285^\circ \quad \text{or } \alpha = 36.176^\circ \quad \checkmark$$

